Non-Parametric Identification and Testing of Quantal Response Equilibrium Online Appendix: Generalizations and Extensions Johannes Hoelzemann Ryan Webb Erhao Xie March 14, 2024

Games with More Players and / or More Actions

This section extends the results in the main text to a general multinomial choice game with $N \ge 2$ players. We use letters *i* and *j* to denote a single player. Letter -i represents all players other than *i*. Each player *i* simultaneously chooses an action, denoted as a_i , from their action set $\mathcal{A}_i = \{0, 1, \dots, K_i\}$. The number of actions (i.e., $K_i + 1 \ge 2$) is unrestricted and could be heterogeneous across players. Moreover, let $\mathbf{a} = (a_i, \mathbf{a}_{-i}) \in$ $\mathcal{A} = \times_{j=1}^N \mathcal{A}_j$ be an action profile of this game, where $\mathbf{a}_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$ is the decision profile made by all players other than *i*. In an experiment, player *i* will receive a monetary payoff $m_i(\mathbf{a})$, in the unit of experimental currency, when **a** is the realized outcome. Consequently, given the utility function $u_i(m)$, player *i* would obtain a utility $u_i[m_i(\mathbf{a})]$ for the profile **a**.

Recall that, as described in the main text, the monetary reward $m_i(\mathbf{a})$ is a control variable in the econometric model. This variable has a support $\mathcal{M}_i(\mathbf{a}) \subset \mathbb{R}$. Furthermore, define \mathbf{m}_i as a $\prod_j (K_j + 1) \times 1$ vector, and each element in this vector represents player *i*'s monetary payoff of the corresponding action profile. Naturally, the vector $\mathbf{m} = (\mathbf{m}'_1, \dots, \mathbf{m}'_N)'$ then summarizes the rewards across profiles and across players. In addition, we use $p_{-i}(\mathbf{a}_{-i}|\mathbf{m})$ to denote the probability that the profile \mathbf{a}_{-i} is chosen by all

players other than *i* given **m**. Similarly, $\mathbf{p}_{-i}(\mathbf{m})$ is a $\prod_{j \neq i} (K_j + 1) \times 1$ vector that consists of the probability of each profile \mathbf{a}_{-i} . With these notations, the function of player *i*'s expected utility for action $a_i = k$ is expressed as:

$$EU_i[\mathbf{m}_i, a_i = k, \mathbf{p}_{-i}(\mathbf{m})] = \sum_{\mathbf{a}_{-i}} u_i[m_i(a_i = k, \mathbf{a}_{-i})] \cdot p_{-i}(\mathbf{a}_{-i}|\mathbf{m}).$$
(46)

In this game with potentially more than two actions, player *i*'s random perturbations extend to a $(K_i+1) \times 1$ vector denoted as $\epsilon_i = (\epsilon_i(0), \epsilon_i(1), \dots, \epsilon_i(K_i))'$. Specifically, this vector of errors follows a joint C.D.F. represented by $\Gamma_i(\epsilon_i)$. Moreover, each element in this vector, denoted as $\epsilon_i(a_i)$, represents player *i*'s calculation error when evaluating the expected utility of the corresponding action. Due to the perturbations of these mistakes, player *i* will choose $a_i = k$ if and only if

$$EU_{i}[\mathbf{m}_{i}, a_{i} = k, \mathbf{p}_{-i}(\mathbf{m})] + \varepsilon_{i}(k) \ge EU_{i}[\mathbf{m}_{i}, a_{i} = k', \mathbf{p}_{-i}(\mathbf{m})] + \varepsilon_{i}(k'), \ \forall k' \neq k.$$
(47)

Define $\mathbf{p}_i(\mathbf{m}) = (p(a_i = 1 | \mathbf{m}), p(a_i = 2 | \mathbf{m}), \dots, p(a_i = K_i | \mathbf{m}))'$ as a $K_i \times 1$ vector that includes player *i*'s choice probability of each action. Note that since the sum of probabilities of all actions equals 1, the choice probability of the base action $a_i = 0$ is suppressed in the vector $\mathbf{p}_i(\mathbf{m})$. Cautiously, our treatment of the vector $\mathbf{p}_{-i}(\mathbf{m})$ is slightly different as this vector consists of the probability of every action profile, including the base profile $\mathbf{a}_{-i} = \mathbf{0}$. This slight distinction in the treatment of $\mathbf{p}_i(\mathbf{m})$ and $\mathbf{p}_{-i}(\mathbf{m})$ simplifies the presentation and proofs of our results.

Under QRE, players' decisions are independent conditional on **m**. Therefore, the joint choice probabilities $\mathbf{p}_{-i}(\mathbf{m})$ for all players other than *i* are determined solely by the individual choice probabilities $\mathbf{p}_j(\mathbf{m})$ for each player $j \neq i$. For instance, $p_{-i}(\mathbf{a}_{-i}|\mathbf{m}) = \prod_{j\neq i} p_j(a_j|\mathbf{m})$. If we interpret QRE as BNE in an incomplete information game where ϵ_i represents player *i*'s private information. The above conditional independence arises from the assumption of independent private information across players (i.e., $\epsilon_i \perp \epsilon_j \forall i \neq j$).

j). In incomplete information games, when $corr(\epsilon_i, \epsilon_j) \neq 0$, player *i*'s private information becomes informative about player *j*'s payoffs and potential decisions. Therefore, each player should adjust their strategies based on their private information, leading to conditional correlated strategies across players. However, within the QRE framework, where ϵ_i is viewed as player *i*'s mistakes rather than private information, it raises an issue for the above channel of correlated actions. Specifically, if player *i*'s strategy depends on the value of ϵ_i , they should be able to distinguish actual utility and calculation errors. However, if such a distinction is clear, player *i* should not make any mistakes. Due to this contradiction, we are not aware of any studies in the framework of QRE that consider ϵ_i to be correlated across players.

In this general multinomial choice game, we modify our assumptions in the 2×2 game as presented in the main text. These modified assumptions are listed below.

Assumption 1'. Each player i's utility function $u_i(m)$ is bounded. Moreover, it is strictly increasing and continuously differentiable in m.

Assumption 2'. For each player *i*, let $\hat{A}(i)$ denote the set of action profiles for which the outcome variable has exogenous variations conditional on player *i*'s outcome variables of other profiles and other players' outcome variables. In other words, $\forall \mathbf{a} \in \hat{A}(i)$, $m_i(\mathbf{a})$ has exogenous variations conditional on $m_i(\mathbf{a}') \ \forall \mathbf{a}' \in \mathcal{A}$ and $\forall \mathbf{m}_{-i} \in \mathcal{M}_{-i}$. We assume that $\hat{A}(i)$ consists of at least two distinct elements and $\cup_{\mathbf{a} \in \hat{A}(i)} \mathcal{M}_i(\mathbf{a}) = \bigcup_{\mathbf{a} \in \mathcal{A}} \mathcal{M}_i(\mathbf{a})$. This union is an interval that could be either open or closed.

Assumption 3'. (a) $\Gamma_i(\epsilon_i)$ has a positive and continuous density function on \mathbb{R}^{K_i+1} , $\forall i$. (b) $\Gamma_i(\epsilon_i)$ is independent of $(\mathbf{m}_i, \mathbf{m}_{-i})$, $\forall i$.

Assumption 4'. For each player *i*, the function of choice probabilities $\mathbf{p}_i(\mathbf{m})$ varies with both \mathbf{m}_i and \mathbf{m}_{-i} . Moreover, $\mathbf{p}_i(\mathbf{m})$ is continuously differentiable for almost every $\mathbf{m} \in \mathcal{M}$. If there are values of \mathbf{m} for which $\mathbf{p}_i(\mathbf{m})$ is not continuously differentiable, the total number of these discontinuous points is finite. Assumption 5'. For each player *i*, there exists a value of the outcome variable, denoted as $m_i^1 \in \bigcup_{\mathbf{a} \in \hat{\mathcal{A}}(i)} \mathcal{M}_i(\mathbf{a})$, such that $u'_i(m_i^1) = 1$.

Assumptions 1' and 4' are slightly stronger than Assumptions 1 and 4. These two modified assumptions not only require the utility function $u_i(m)$ and the choice probability function $\mathbf{p}_i(\mathbf{m})$ to be continuous but also to be differentiable. This differentiation simplifies the proofs in this general multinomial choice game with $N \ge 2$ players. Moreover, due to the expanded space of action profiles, Assumption 2' requires the analyst to exogenously vary the monetary payoffs of at least two action profiles within each player. This is in contrast to most of identification results in 2×2 games, where the exogenous variation of a single profile's payoff suffices.

Assumption 3'(a) and Assumption 3'(b) are standard regularity and invariance conditions for the error distributions, adapted for games with more actions. These conditions allow for general error structures. Specifically, the error of each action could follow a heterogeneous marginal distribution and exhibit arbitrary correlation with the error of another action.

Assumption 5' is an alternative but equivalent scale normalization compared to Assumption 5. Specifically, consider the affine transformation $u_i(m) = c + \beta \hat{u}_i(m)$, Assumption 5' simply transforms $\hat{u}_i(m)$ to its equivalent form by setting $\beta = \frac{1}{\hat{u}'_i(m_i^1)}$. Since most of the proofs in this generalization are based on derivatives, it is convenient to normalize the marginal utility as in Assumption 5'.

In discrete choice models, the decision rule by Equation (47) implies the following mapping between player *i*'s expected utility differences and their choice probabilities:

$$\mathbf{p}_i(\mathbf{m}) = \mathbf{F}_i[\mathbf{E}\mathbf{U}_i(\mathbf{m}_i, \mathbf{p}_{-i}(\mathbf{m}))], \qquad (48)$$

where $\widetilde{\mathbf{EU}}_i(\mathbf{m}_i, \mathbf{p}_{-i}(\mathbf{m}))$ is a $K_i \times 1$ vector that represents the difference in expected utilities for player *i*. In particular, the k^{th} element of this vector, denoted as $\widetilde{EU}_i[\mathbf{m}_i, a_i = k, \mathbf{p}_{-i}(\mathbf{m})]$, is defined as $EU_i[\mathbf{m}_i, a_i = k, \mathbf{p}_{-i}(\mathbf{m})] - EU_i[\mathbf{m}_i, a_i = 0, \mathbf{p}_{-i}(\mathbf{m})]$. As standard in discrete choice models, these differences of expected utilities completely determine player *i*'s choice probabilities (Train, 2009). This relationship can be represented by the mapping $\mathbf{F}_i : \mathbb{R}^{K_i} \to int(\Delta^{K_i})$, where $int(\Delta^{K_i})$ denotes the interior of K_i -dimensional simplex. The k^{th} element in this mapping, denoted as $F_{i,k}(\cdot)$, then represents the resulting choice probability of action $a_i = k$. Under Assumption 3'(a), the mapping $\mathbf{F}_i(\cdot)$ is bijective (Norets and Takahashi, 2013; Sørensen and Fosgerau, 2022). Moreover, Hotz and Miller (1993) show that $\mathbf{F}_i(\cdot)$ is differentiable.

In this general multinomial choice game with $N \ge 2$ players, QRE is defined by a fixed-point condition, as summarized by Definition 1'.

Definition 1'. The vector $(\mathbf{p}_1(\mathbf{m})', \mathbf{p}_2(\mathbf{m})', \cdots, \mathbf{p}_N(\mathbf{m})')$ denotes the QRE choice probabilities if and only if the following condition holds:

$$\mathbf{p}_{i}(\mathbf{m}) = \mathbf{F}_{i}[\mathbf{E}\mathbf{U}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))], \forall i \text{ and } \mathbf{m} \in \mathcal{M}.$$
(49)

For any $\mathbf{m} \in \mathcal{M}$, if there are multiple vectors that satisfy Equation (49) (i.e., multiple *QRE*), there exists a mechanism that selects one of the vectors / equilibria.

In this section, we first prove that player *i*'s utility function is non-parametrically identified. We then exploit this result to show the over-identification of $\mathbf{F}_i(\cdot)$ and the testable implication of QRE.

Proposition 7. Suppose that Assumptions 1' to 5' and QRE restrictions hold, then the derivative of the utility function $u'_i(m)$ is point identified $\forall m \in \bigcup_{\mathbf{a}} \mathcal{M}_i(\mathbf{a})$ and $\forall i$.

Proof. Under Assumption 2', let $\mathbf{a}' = (a'_i, a'_{-i}) \neq \mathbf{a}'' = (a''_i, a''_{-i})$ be the two action profiles in the set $\hat{\mathcal{A}}(i)$. We assume that $a'_i \neq 0$ and $a''_i \neq 0$. This is without loss of generality since the analyst could always re-label player *i*'s actions. As described above, Assumptions 1', 3' and 4' imply that every function in Equation (49) is differentiable with respect to their arguments. Consequently, we could take derivative on both sides of Equation (49) and obtain the following relationship:

$$\frac{\partial \mathbf{p}_{i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}')} = \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \frac{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial m_{i}(\mathbf{a}')} + \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \widetilde{\mathbf{I}}_{i}(\mathbf{m}_{i}) \frac{\partial \mathbf{p}_{-i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}')}, \\
\frac{\partial \mathbf{p}_{i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}'')} = \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \frac{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial m_{i}(\mathbf{a}'')} + \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \widetilde{\mathbf{I}}_{i}(\mathbf{m}_{i}) \frac{\partial \mathbf{p}_{-i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}'')}.$$
(50)

Note that $\tilde{\Pi}_i(\mathbf{m}_i)$ is a $K_i \times \prod_{j \neq i} (K_j + 1)$ matrix whose element in cell (a_i, \mathbf{a}_{-i}) is represented by $\tilde{\pi}_i(\mathbf{m}_i, a_i, \mathbf{a}_{-i}) = u_i[m_i(a_i, \mathbf{a}_{-i})] - u_i[m_i(a_i = 0, \mathbf{a}_{-i})].$

Equation (49) suggests that $\mathbf{p}_i(\mathbf{m})$ could be alternatively represented as $\mathbf{p}_i(\mathbf{m}_i, \mathbf{p}_{-i}(\mathbf{m}))$. This equivalent form could be consistently estimated from choice data, due to the consistent estimation of $\mathbf{p}_{-i}(\mathbf{m})$ as described in the main text. Consequently, we could take derivative for this equivalent form and obtain:

$$\frac{\partial \mathbf{p}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))}{\partial \mathbf{p}_{-i}'(\mathbf{m})} = \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))} \widetilde{\mathbf{\Pi}}_{i}(\mathbf{m}_{i}).$$
(51)

Substituting Equation (51) into Equation (50) leads to:

$$\frac{\partial \mathbf{p}_{i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}')} - \frac{\partial \mathbf{p}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial \mathbf{p}'_{-i}(\mathbf{m})} \frac{\partial \mathbf{p}_{-i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}')} = \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \frac{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial m_{i}(\mathbf{a}')},$$

$$\frac{\partial \mathbf{p}_{i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}'')} - \frac{\partial \mathbf{p}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial \mathbf{p}'_{-i}(\mathbf{m})} \frac{\partial \mathbf{p}_{-i}(\mathbf{m})}{\partial m_{i}(\mathbf{a}'')} = \frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))} \frac{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m}))}{\partial m_{i}(\mathbf{a}'')}.$$
(52)

Since each player's choice probability can be consistently estimated, the terms on the left-hand side of Equation (52) are known to the analyst. Consequently, the terms on the right-hand side are identified.

Consider an arbitrary $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ where $a_i \neq 0$. The structure of expected utilities

implies the following expression:

$$\frac{\partial \mathbf{F}_{i}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))} \frac{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m}))}{\partial m_{i}(\mathbf{a})} = \begin{pmatrix} \frac{\partial F_{i,1}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},\mathbf{p}_{-i}(\mathbf{m})]]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},a_{i},\mathbf{p}_{-i}(\mathbf{m}))} u_{i}'[m_{i}(\mathbf{a})]p_{-i}(\mathbf{a}_{-i}|\mathbf{m}) \\ \frac{\partial F_{i,2}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},a_{i},\mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},a_{i},\mathbf{p}_{-i}(\mathbf{m}))} u_{i}'[m_{i}(\mathbf{a})]p_{-i}(\mathbf{a}_{-i}|\mathbf{m}) \\ \vdots \\ \frac{\partial F_{i,K_{i}}[\widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},a_{i},\mathbf{p}_{-i}(\mathbf{m}))]}{\partial \widetilde{\mathbf{EU}}_{i}(\mathbf{m}_{i},a_{i},\mathbf{p}_{-i}(\mathbf{m}))} u_{i}'[m_{i}(\mathbf{a})]p_{-i}(\mathbf{a}_{-i}|\mathbf{m}) \end{pmatrix}$$
(53)

Substituting Equation (53) into Equation (52) implies that the terms $\frac{\partial F_{i,k}(\widetilde{\mathbf{EU}}_i)}{\partial \widetilde{EU}_i(\cdot,a'_i)}u'_i[m_i(\mathbf{a}')]p_{-i}(\mathbf{a}'_{-i}|\mathbf{m})$ and $\frac{\partial F_{i,k}(\widetilde{\mathbf{EU}}_i)}{\partial \widetilde{EU}_i(\cdot,a''_i)}u'_i[m_i(\mathbf{a}'')]p_{-i}(\mathbf{a}''_{-i}|\mathbf{m})$ are identified for each *k*. It further implies the following result:

$$\frac{\frac{\partial F_{i,a_i''}[\mathbf{E}\mathbf{U}_i(\mathbf{m}_i,\mathbf{p}_{-i}(\mathbf{m}))]}{\partial E U_i(\mathbf{m}_i,a_i',\mathbf{p}_{-i}(\mathbf{m}))} u_i'[m_i(\mathbf{a}')]p_{-i}(\mathbf{a}'_{-i}|\mathbf{m})}{\frac{\partial F_{i,a_i'}[\mathbf{E}\mathbf{U}_i(\mathbf{m}_i,\mathbf{p}_{-i}(\mathbf{m}))]}{\partial E U_i(\mathbf{m}_i,a_i'',\mathbf{p}_{-i}(\mathbf{m}))}} u_i'[m_i(\mathbf{a}'')]p_{-i}(\mathbf{a}''_{-i}|\mathbf{m})} = \frac{u_i'[m_i(\mathbf{a}')]p_{-i}(\mathbf{a}'_{-i}|\mathbf{m})}{u_i'[m_i(\mathbf{a}'')]p_{-i}(\mathbf{a}''_{-i}|\mathbf{m})} \text{ is identified.}$$
(54)

The equality of Equation (54) follows the results in discrete choice models such that $\frac{\partial \mathbf{F}_i(\widetilde{\mathbf{EU}}_i)}{\partial \widetilde{\mathbf{EU}}_i(\cdot)}$ can be seen as the hessian matrix of the social welfare function (Sørensen and Fosgerau, 2022). The social welfare function is strictly convex. Therefore, the matrix $\frac{\partial \mathbf{F}_i(\widetilde{\mathbf{EU}}_i)}{\partial \widetilde{\mathbf{EU}}_i(\cdot)}$ is symmetric and positive definite. This symmetry implies that $\frac{\partial F_{i,a_i''}(\cdot)}{\partial \widetilde{\mathbf{EU}}_i(\cdot,a_i')} = \frac{\partial F_{i,a_i'}(\cdot)}{\partial \widetilde{\mathbf{EU}}_i(\cdot,a_i'')}$ and can be cancelled out in Equation (54). Moreover, each term in the denominator is strictly positive so that the ratio is well defined. In particular, $\frac{\partial F_{i,a_i'}(\cdot)}{\partial \widetilde{\mathbf{EU}}_i(\cdot,a_i'')} > 0$ due to the strictly positive density in Assumption 3'(a); $u_i'(\cdot) > 0$ due to the strict monotonicity as per Assumption 1'; $p_{-i}(\mathbf{a}''_{-i}|\mathbf{m}) > 0$ due to the full support condition of ϵ_i in Assumption 3'(a).

Equation (54) identifies $\frac{u'_i(m')}{u'_i(m'')}$ for any m', $m'' \in \bigcup_{\mathbf{a} \in \hat{\mathcal{A}}(i)} \mathcal{M}_i(\mathbf{a}) = \bigcup_{\mathbf{a} \in \mathcal{A}} \mathcal{M}_i(\mathbf{a})$. Assumption 5' normalizes the scale of $u'_i(m^1_i)$ at one arbitrary value m^1_i . This normalization then identifies $u'_i(m) \ \forall m \in \bigcup_{\mathbf{a} \in \mathcal{A}} \mathcal{M}_i(\mathbf{a})$ and completes the proof.

Proposition 7 identifies the marginal utility $u'_i(m)$ and consequently identifies the utility function in the class of $u_i(m) + c$. As in the main text, when the analyst considers

the normalization, such as $u_i(0) = 0$ or $u_i[\min\{\bigcup_{\mathbf{a}} \mathcal{M}_i(\mathbf{a})\}] = 0$, the value of *c* is identified. It further identifies the utility function $u_i(m) \ \forall m \in \bigcup_{\mathbf{a}} \mathcal{M}_i(\mathbf{a})$.

As described above, Assumption 3'(a) implies that the mapping $\mathbf{F}_i(\cdot)$ is bijective (Norets and Takahashi, 2013; Sørensen and Fosgerau, 2022). Therefore, we could invert $\mathbf{F}_i(\cdot)$ and the QRE restriction by Equation (49) becomes:

$$\mathbf{F}_{i}^{-1}[\mathbf{p}_{i}(\mathbf{m})] = \widetilde{\mathbf{EU}}_{i}[\mathbf{m}_{i}, \mathbf{p}_{-i}(\mathbf{m})] = \widetilde{\mathbf{\Pi}}_{i}(\mathbf{m}_{i}) \cdot \mathbf{p}_{-i}(\mathbf{m}).$$
(55)

This equation implies the non-parametric identification of the error distribution, as established by Proposition 1'.

Proposition 1'. Suppose that Assumptions 1' to 5' and QRE restrictions hold; therefore, the marginal utility $u'_i(m)$ is point identified for each player i by Proposition 7. In the next step, suppose that the analyst fixes \mathbf{m}_i at an arbitrary value \mathbf{m}_i^1 and only considers the variation of \mathbf{m}_{-i} , then $\mathbf{F}_i^{-1}(\mathbf{p})$ is point identified $\forall \mathbf{p} \in \mathcal{P}_i(\mathbf{m}_i^1)$.

Proof. As described above, the identification of $u'_i(m)$ implies that the utility function is identified in the class of $u_i(m) + c$. Consequently, the difference of utilities $\tilde{\pi}_i(\mathbf{m}_i, a_i, \mathbf{a}_{-i}) = u_i[m_i(a_i, \mathbf{a}_{-i})] - u_i[m_i(a_i = 0, \mathbf{a}_{-i})]$ is uniquely determined as the constant c is cancelled out. It further implies that the matrix $\tilde{\mathbf{\Pi}}_i(\mathbf{m}_i)$ is known to the analyst for each $\mathbf{m}_i \in \mathcal{M}_i$. For an arbitrary value \mathbf{m}_i^1 , Equation (55) turns to:

$$\mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i^1, \mathbf{m}_{-i})] = \tilde{\mathbf{\Pi}}_i(\mathbf{m}_i^1) \cdot \mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}).$$
(56)

The terms on the right-hand side are known to the analyst. Consequently, the exogenous variation of \mathbf{m}_{-i} then identifies $\mathbf{F}_i^{-1}(\mathbf{p})$ for all values of \mathbf{p} in the support of $\mathcal{P}_i(\mathbf{m}_i^1)$. This completes the proof.

Due to the inverse relationship between $\mathbf{F}_i^{-1}(\cdot)$ and $\mathbf{F}_i(\cdot)$, Proposition 1' implies the non-parametric identification of the mapping $\mathbf{F}_i(\cdot)$. It further implies that the distribution

of the difference of errors $\tilde{\boldsymbol{\epsilon}}_i = (\boldsymbol{\epsilon}_i(1) - \boldsymbol{\epsilon}_i(0), \boldsymbol{\epsilon}_i(2) - \boldsymbol{\epsilon}_i(0), \cdots, \boldsymbol{\epsilon}_i(K_i) - \boldsymbol{\epsilon}_i(0))'$ is uniquely determined (Train, 2009).

Recall that $\hat{\mathbf{F}}_i^{-1}(\mathbf{p}|\mathbf{m}_i^1)$ represents the inverted choice probability function that satisfies the QRE restrictions when the analyst fixes \mathbf{m}_i at \mathbf{m}_i^1 . Proposition 1' directly implies a testable restriction of QRE, as established below.

Proposition 2'. Suppose that Assumptions 1' to 5' and QRE restrictions hold. Consider any two realizations of \mathbf{m}_i , denoted as \mathbf{m}_i^1 and \mathbf{m}_i^2 . Suppose that $\mathcal{P}_i(\mathbf{m}_i^1) \cap \mathcal{P}_i(\mathbf{m}_i^2) \neq \emptyset$, then the null hypothesis of QRE implies the following testable restriction:

$$\hat{\mathbf{F}}_{i}^{-1}(\mathbf{p}|\mathbf{m}_{i}^{1}) = \hat{\mathbf{F}}_{i}^{-1}(\mathbf{p}|\mathbf{m}_{i}^{2}), \ \forall \mathbf{p} \in \mathcal{P}_{i}(\mathbf{m}_{i}^{1}) \cap \mathcal{P}_{i}(\mathbf{m}_{i}^{2}).$$
(57)

Proof. A direct implication of Proposition 1'.

Comparison to Xie (2022) In multinomial choice games with field data, Xie (2022) establishes the non-parametric identification of the utility and the error distribution. However, to prove these results, he imposes two strong restrictions: one on the model primitive and the other on the data. In contrast, this paper shows that in experimental datasets where the outcome variable is observed, the non-parametric identification results can be achieved without imposing these two strong restrictions. We elaborate these two restrictions below.

The first restriction in Xie (2022) is that the error distribution must satisfy a *rank ordering property*. Under this property, one action is chosen more frequently than another if and only if it yields a strictly higher expected utility. While this assumption is often made in empirical applications of QRE, it imposes strong restrictions on the error distribution. In particular, this property rules out error structures with flexible correlations and features of heteroskedasticity. To illustrate this point, consider an agent facing three actions labeled as 1, 2, and 3, with associated expected utilities denoted as

EU(1), EU(2), and EU(3). Without loss of generality, suppose that EU(1) > EU(2) >EU(3). Consider one structure where the errors $\varepsilon(k)$ follow the same marginal distribution but can be correlated across actions. In particular, $\varepsilon(1)$ and $\varepsilon(2)$ exhibit strong positive correlation, and they are independent of $\varepsilon(3)$. This error structure implies that $EU(1) + \varepsilon(1) > EU(2) + \varepsilon(2)$ would hold with high probability, leading to a low choice probability of action 2. However, action 3 might be chosen more frequently than action 2 due to its independent error $\varepsilon(3)$. Next, consider another structure where $\varepsilon(k)$ is independent across actions but are heterogeneous in their scales. In particular, the agent only makes minor mistakes for the first two actions, resulting in small $Var(\varepsilon(1))$ and $Var(\varepsilon(2))$. Consequently, action 2 is still chosen with a small probability since its perturbed expected utility is likely to be smaller than that of action 1. In contrast, suppose $\varepsilon(3)$ has a large variance, then the third action could be chosen more frequently than action 2 since this agent may frequently and mistakenly evaluate action 3 as highly attractive. In summary, under these two reasonable error structures, the agent will choose action 3 more frequently than action 2 even though action 3 has a lower expected utility. Clearly, the rank ordering property is violated.

Xie (2022) also considers another strong restriction on the data. In particular, he defines two actions to be *connected* if these two actions can be chosen with equal probability. His identification results require each pair of two actions to be either connected or linked through a sequence of connected actions. This particular data structure is challenging to construct in an experiment.

Implications of Assumption 2'

This section focuses on 2×2 games and discusses the implications of Assumption 2' on the results presented in the main text. First, under this assumption, the previous section establishes the identification results without assuming prior knowledge of the values of $F_i^{-1}(p^1)$ and $F_i^{-1}(p^2)$. Consequently, when player *i*'s monetary payoffs exhibit

exogenous variations for at least two action profiles, the analyst does not need to impose the payoff structures as required by Assumption 6 to identify the median and the mean of the errors.

Assumption 2' itself also implies that our over-identification test incorporates more restrictions than the test by Xie (2022). Specifically, let $\mathbf{a}' \neq \mathbf{a}''$ be the two action profiles in $\hat{\mathcal{A}}(i)$ as required by Assumption 2'. When $\mathcal{M}_i(\mathbf{a}') \cap \mathcal{M}_i(\mathbf{a}'')$ is an interval, there exist infinite pairs where each pair contains two values \mathbf{m}_i^1 and \mathbf{m}_i^2 such that $m_i^1(\mathbf{a}') = m_i^2(\mathbf{a}'')$ and $m_i^2(\mathbf{a}') = m_i^1(\mathbf{a}'')$. Moreover, between \mathbf{m}_i^1 and \mathbf{m}_i^2 , let player *i*'s monetary rewards hold constant for all action profiles other than \mathbf{a}' and \mathbf{a}'' . We claim that these pairs imply testable restrictions of QRE in addition to Xie (2022).

Consider the first scenario that $a'_{-i} \neq a''_{-i}$. The pair \mathbf{m}_i^1 and \mathbf{m}_i^2 implies that $\tilde{\pi}_i(\mathbf{m}_i^1, a'_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^2, a'_{-i}) = \tilde{\pi}_i(\mathbf{m}_i^1, a'_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^2, a''_{-i}) = \tilde{\pi}_i(\mathbf{m}_i^2, a''_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^2, a'_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^2, a'_{-i}) = \tilde{\pi}_i(\mathbf{m}_i^2, a''_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^1, a'_{-i}) = \tilde{\pi}_i(\mathbf{m}_i^2, a''_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^1, a'_{-i}) = \tilde{\pi}_i(\mathbf{m}_i^2, a''_{-i})$. The pair \mathbf{m}_i^1 and \mathbf{m}_i^2 then implies that $\tilde{\pi}_i(\mathbf{m}_i^1, a'_{-i}) = -\tilde{\pi}_i(\mathbf{m}_i^2, a'_{-i})$. These linear restrictions on utility differences $\tilde{\pi}_i(\cdot)$ lead to additional linear restrictions on $\hat{F}_i^{-1}(p|\mathbf{m}_i)$ through the definition of $\hat{F}_i^{-1}(p|\mathbf{m})$ by Equation (6). These restrictions are therefore included in our overidentification test that is based on $\hat{F}_i^{-1}(p|\mathbf{m}_i)$. However, they are excluded from the test by Xie (2022).

Relaxing the Invariance Assumption When there are at least two action profiles with varying monetary payoffs as per Assumption 2', the analyst is able to relax the invariance condition as in Assumption 3(b) and still tests the null hypothesis of QRE. In particular, we allow player *i*'s distribution function $F_i(\tilde{\epsilon}_i)$ to depend on their own rewards \mathbf{m}_i and restrict it to be independent of other players' payoffs \mathbf{m}_{-i} . To capture this dependence, we adapt the notation $F_i(\tilde{\epsilon}_i | \mathbf{m}_i)$.

Let \mathbf{a}' and \mathbf{a}'' be the two distinct profiles in $\hat{\mathcal{A}}(-i)$. Suppose that the analyst fixes $m_{-i}(\mathbf{a}') = m_{-i}^1$ and varies player -i's payoffs for the other profile \mathbf{a}'' . In alignment with the main text, we denote $\hat{F}_i^{-1}(p|\mathbf{m}_i, m_{-i}(\mathbf{a}') = m_{-i}^1)$ as the quantile function that satisfies

QRE restrictions when $m_{-i}(\mathbf{a}')$ is fixed at m_{-i}^1 . The proof of Proposition 1 indicates that this quantile function is identified for any value of $m_{-i}(\mathbf{a}')$ based on the variation provided by $m_{-i}(\mathbf{a}'')$. Consequently, the null hypothesis of QRE implies the following testable implication:

$$\hat{F}_{i}^{-1}(p|\mathbf{m}_{i},m_{-i}(\mathbf{a}')=m_{-i}^{1})=\hat{F}_{i}^{-1}(p|\mathbf{m}_{i},m_{-i}(\mathbf{a}')=m_{-i}^{2}),\ \forall \mathbf{m}_{i},m_{-i}^{1},m_{-i}^{2}$$