Non-Parametric Identification and Testing of Quantal Response Equilibrium

Online Appendix: Generalizations and Extensions

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Games with More Players and/or More Actions

This section generalizes the results in the main text to a *general multinomial choice game*. This game has $N \geq 2$ players and each player could have any finite number of possible choices. In particular, we use letters i and j to denote two arbitrary players. Letter $-i$ represents all players other than i. Each player i simultaneously chooses an action, denoted by a_i, from their action set $A_i = \{0, 1, \cdots, K_i\}$. Therefore, player i has $(K_i + 1)$ possible alternatives. Moreover, let $\mathbf{a} = (a_i, \mathbf{a}_{-i}) \in A = \times_{j=1}^{N} A_j$ denote an action profile of this game, where $\mathbf{a}_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_N)$ represents the decision profile made by all players other than i. Then player i's utility of the profile $\mathbf{a}$ is $\pi_i(\mathbf{m}_i, \mathbf{a})$. Finally, denote $p_{-i}(\mathbf{a}_{-i}|\mathbf{m})$ as the probability that the profile $\mathbf{a}_{-i}$ is chosen by all players other than i, given $\mathbf{m} = (\mathbf{m}_1', \cdots, \mathbf{m}_N')'$; then player i's expected utility of action $a_i = k$ is

$$\begin{aligned}
&E\pi_i[\mathbf{m}_i, a_i = k, \mathbf{p}_{-i}(\mathbf{m})] \\
=&\sum_{\mathbf{a}_{-i}} \pi_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i}) \cdot p_{-i}(\mathbf{a}_{-i}|\mathbf{m}) \qquad (23)\\
=&\pi_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = \mathbf{0}) + \sum_{\mathbf{a}_{-i} \neq \mathbf{0}} [\pi_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i}) - \pi_i(\mathbf{m}_i, a_i = k, \mathbf{0})] \cdot p_{-i}(\mathbf{a}_{-i}|\mathbf{m}).
\end{aligned}$$

Specifically, $\mathbf{0}$ is an $(N-1)\times 1$ vector. It represents the event that all players other than i choose action 0. Moreover, $\mathbf{p}_{-i}(\cdot)$ is a $\left(\prod_{j\neq i}(K_j+1)-1\right)\times 1$ vector. Each element in this vector denotes the probability of the corresponding action profile chosen by player $-i$. Note that since the sum of all profiles' probabilities equals 1, we exclude the event $\mathbf{a}_{-i}=\mathbf{0}$ in $\mathbf{p}_{-i}(\cdot)$ for simplicity.

In this game with potentially more than two actions, player i's random perturbation extends to a $(K_i+1)\times 1$ vector, denoted by $\boldsymbol{\epsilon}_i=(\varepsilon_i(0),\varepsilon_i(1),\cdots,\varepsilon_i(K_i))'$. Note that $\varepsilon_i(a_i=k)$ represents player i's calculation error of action k. Due to the perturbation of these mistakes, player i will choose action k if and only if

$$E\pi_i[\mathbf{m}_i,a_i=k,\mathbf{p}_{-i}(\mathbf{m})]+\varepsilon_i(k)\geq E\pi_i[\mathbf{m}_i,a_i=k',\mathbf{p}_{-i}(\mathbf{m})]+\varepsilon_i(k'),\ \forall k'\neq k. \tag{24}$$

Let $\Gamma_i(\boldsymbol{\epsilon}_i)$ denote the C.D.F. of $\boldsymbol{\epsilon}_i=(\varepsilon_i(0),\varepsilon_i(1),\cdots,\varepsilon_i(K_i))'$. Note that $\varepsilon_i(a_i=k)$. We focus on the regular case that $\Gamma_i(\boldsymbol{\epsilon}_i)$ is absolutely continuous with respect to the Lebesgue measure. Put differently, $\Gamma_i(\boldsymbol{\epsilon}_i)$ has a density with respect to the Lebesgue measure. When there are only two actions, the above condition reduces to the strict monotonicity of $F_i(\cdot)$, as required in the main text.

Assumption 2' simply modifies the invariance Assumption 2 to accommodate the new notation.

Assumption 2'. *$\Gamma_i(\cdot)$ is independent of $(\mathbf{m}_i,\mathbf{m}_{-i})$.*

Given $\Gamma_i(\cdot)$, player i's choice probability of action k – denoted by $p_i(a_i=k|\mathbf{m})$ – takes the following form:

$$\begin{aligned}&p_i(a_i=k|\mathbf{m})\\=&\int_{\boldsymbol{\epsilon}_i}\mathbb{1}\left\{E\pi_i[\mathbf{m}_i,k,\mathbf{p}_{-i}(\mathbf{m})]+\varepsilon_i(k)\geq E\pi_i[\mathbf{m}_i,k',\mathbf{p}_{-i}(\mathbf{m})]+\varepsilon_i(k'),\ \forall k'\neq k\right\}d\Gamma_i(\boldsymbol{\epsilon}_i)\\=&F_{i,k}\left\{\boldsymbol{E\tilde{\pi}}_i[\mathbf{m}_i,\mathbf{p}_{-i}(\mathbf{m})]\right\}.\end{aligned} \tag{25}$$

Naturally, $F_{i,k}(\cdot)$ denotes the k^{th} element of the integral function that maps player i's expected utilities to their choice probabilities. Moreover, $\boldsymbol{E\tilde{\pi}}_i(\cdot) = \big(E\pi_i(\cdot, a_i = 1) - E\pi_i(\cdot, a_i = 0), E\pi_i(\cdot, a_i = 2) - E\pi_i(\cdot, a_i = 0), \cdots, E\pi_i(\cdot, a_i = K_i) - E\pi_i(\cdot, a_i = 0)\big)'$ is a $K_i \times 1$ vector. It represents the difference of each action's expected utility with respect to the base action, denoted by action 0. As standard in the discrete choice literature (Train, 2009), it is this difference that completely determines a player's choice probability. Therefore, our analysis focuses on $\boldsymbol{E\tilde{\pi}}_i(\cdot)$. Finally, let $\mathbf{p}_i(\mathbf{m}) = \big(p_i(a_i = 1|\mathbf{m}), p_i(a_i = 2|\mathbf{m}), \cdots, p_i(a_i = K_i|\mathbf{m})\big)'$ be a $K_i \times 1$ vector that consists of player i's choice probability of each action. Again, since the probabilities of all actions sum to 1, action 0 is excluded from $\mathbf{p}_i(\cdot)$ for simplicity. It is convenient to represent Equation (25) in the following matrix form:

$$\mathbf{p}_i(\mathbf{m}) = \mathbf{F}_i\big\{\boldsymbol{E\tilde{\pi}}_i[\mathbf{m}, \mathbf{p}_{-i}(\mathbf{m})]\big\}. \tag{26}$$

Similar as Definition 1 in the main text, QRE imposes a fixed-point condition such that Equation (26) holds for every player. It is summarized by Definition 1'.

Definition 1'. *The vector* $\mathbf{p}(\mathbf{m}) = (\mathbf{p}_1(\mathbf{m})', \mathbf{p}_2(\mathbf{m})', \cdots, \mathbf{p}_N(\mathbf{m})')'$ *denotes the QRE choice probabilities if and only if Equation (26) is satisfied* $\forall 1 \leq i \leq N,\ \mathbf{m}$.

Hotz and Miller (1993) and Norets and Takahashi (2013) show that – under the absolutely continuous condition of $\Gamma_i(\cdot)$ – the mapping from player i's expected utilities to their choice probabilities is bijective. Consequently, function $\mathbf{F}_i(\cdot)$ is invertible. Inverting Equation (26) yields the following equation:

$$\boldsymbol{E\tilde{\pi}}_i[\mathbf{m}_i, \mathbf{p}_{-i}(\mathbf{m})] = \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m})]$$

or

$$\boldsymbol{\tilde{\pi}}_i(\mathbf{m}_i) + \boldsymbol{\Delta}_i(\mathbf{m}_i) \cdot \mathbf{p}_{-i}(\mathbf{m}) = \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m})]. \tag{27}$$

The second line of Equation (27) decomposes $\boldsymbol{E\tilde{\pi}}_i(\cdot)$ into two parts: $\boldsymbol{\tilde{\pi}}_i(\cdot)$ and $\boldsymbol{\Delta}_i(\cdot) \cdot$

$\mathbf{p}_{-i}(\cdot)$. Term $\tilde{\pi}_i(\cdot)$ is a $K_i \times 1$ vector. Its k^{th} element is the utility difference between action k and action 0, when all other players choose the base action 0. For instance, it is $\tilde{\pi}_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = \mathbf{0}) = \pi_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = \mathbf{0}) - \pi_i(\mathbf{m}_i, a_i = 0, \mathbf{a}_{-i} = \mathbf{0})$. In the literature that estimates empirical games, this term is referred to as the *base return*. In addition, $\mathbf{\Delta}_i(\cdot)$ is a $K_i \times \left(\prod_{j \neq i}(K_j + 1) - 1\right)$ matrix. The element on the k^{th} row and l^{th} column of this matrix is $\delta_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = l) = \tilde{\pi}_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = l) - \tilde{\pi}_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i} = \mathbf{0})$. Specifically, it represents the change of the utility difference when other players deviate their behaviors from $\mathbf{0}$ to the action profile indexed by l. Naturally, this term is referred to as the *strategic interaction*. For convenience, we assume that the matrix $\mathbf{\Delta}_i(\cdot)$ has a full rank. This assumption is extremely weak. To see this point, note that $\mathbf{\Delta}_i(\mathbf{m}_i)$ is random due to the random variables $\mathbf{m}_i$. Consequently, $\mathbf{\Delta}_i$ will have a full rank with probability 1. This is because in the pace of all possible matrices, the set of full rank matrices has a full measure. Importantly, when $\mathbf{\Delta}_i(\cdot)$ is rank deficient, the testable implication of QRE still exists but just turns to be slightly weaker.

The decomposition in the second line of Equation (27) follows the definition of $\boldsymbol{E}\tilde{\pi}_i(\cdot)$. It separates the utility function (i.e., $\tilde{\pi}_i(\cdot)$ and $\mathbf{\Delta}_i(\cdot)$) and other players' choice probabilities $\mathbf{p}_{-i}(\mathbf{m})$. This decomposition is convenient to prove the main results. Similar as the main text, Equation (27) contains all model restrictions that are imposed on player i's behaviors. It directly leads a testable implication of QRE. To see this implication, define $\tilde{\mathbf{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}) = \mathbf{p}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^2) - \mathbf{p}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^1)$ and $\tilde{\mathbf{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}) = \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}_{-i}^2)] - \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}_{-i}^1)]$ as the differences of player $-i$'s choice probabilities and inverted distribution function between two realizations of $\mathbf{m}_{-i}$, denoted by $\mathbf{m}_{-i}^1$ and $\mathbf{m}_{-i}^2$. Further, for any $H > 2$, let $\tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H}) = \left(\tilde{\mathbf{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}), \cdots, \tilde{\mathbf{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,H})\right)'$ and $\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H}) = \left(\tilde{\mathbf{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}), \cdots, \tilde{\mathbf{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,H})\right)'$ be the corresponding matrices. With the above notations, Proposition 1' presents a testable implication of QRE in this general multinomial choice game.

Proposition 1'. *Under Assumptions 1 and 2', for any* $H = \prod_{j \neq i}(K_j + 1) + 1$ *pairs of* *realizations of* $(\mathbf{m}_i, \mathbf{m}_{-i})$ *– denoted by* $(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(h)})$ *and* $(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(h)})$ $\forall h \leq H$ *– that satisfy the following condition of equal choice probability:*

$$\mathbf{p}_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(h)}) = \mathbf{p}_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(h)}),\ \forall h \leq H.$$

Given these pairs, QRE implies the following testable restriction:

$$\begin{aligned}
&Rank\big[\tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2:H)}) - \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2:H)})\big] \\
\leq &\prod_{j\neq i}(K_j+1) - 1 - min\big(K_i, \prod_{j\neq i}(K_j+1) - 1\big). \qquad (28)
\end{aligned}$$

Note that the above is a $\big(\prod_{j\neq i}(K_j+1)-1\big) \times \big(\prod_{j\neq i}(K_j+1)-1\big)$ *matrix.*

Proof. Consider two realizations of $\mathbf{m}_{-i}$, denoted by $\mathbf{m}^1_{-i}$ and $\mathbf{m}^2_{-i}$, and plug them into Equation (27). It then obtains the following expressions:

$$\begin{aligned}
\tilde{\boldsymbol{\pi}}_i(\mathbf{m}_i) + \boldsymbol{\Delta}_i(\mathbf{m}_i) \cdot \mathbf{p}_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i}) &= \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^1_{-i})], \\
\tilde{\boldsymbol{\pi}}_i(\mathbf{m}_i) + \boldsymbol{\Delta}_i(\mathbf{m}_i) \cdot \mathbf{p}_{-i}(\mathbf{m}_i, \mathbf{m}^2_{-i}) &= \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^2_{-i})].
\end{aligned}$$

Subtracting these two equations implies the following difference of expected utilities:

$$\boldsymbol{\Delta}_i(\mathbf{m}_i) \cdot \tilde{\mathbf{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}) = \tilde{\mathbf{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1,2}). \qquad (29)$$

Additionally, if we consider realizations $\mathbf{m}^1_{-i}$ to $\mathbf{m}_{-i}^{H-1}$ of $\mathbf{m}_{-i}$, Equation (29) could be expressed in the following matrix form:

$$\begin{aligned}
&\boldsymbol{\Delta}_i(\mathbf{m}_i) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) = \tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) \\
\Rightarrow &\boldsymbol{\Delta}_i(\mathbf{m}_i) = \tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}). \qquad (30)
\end{aligned}$$

Analogously, realizations $\mathbf{m}^2_{-i}$ to $\mathbf{m}^H_{-i}$ of $\mathbf{m}_{-i}$ would imply a similar relationship

$$\Delta_i(\mathbf{m}_i) = \tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}). \tag{31}$$

Given that the term $\Delta_i(\mathbf{m}_i)$ on the left hand sides of Equations (30) and (31) is the same, equalizing these two equations implies the following:

$$\begin{aligned}
&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) = \tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}) \\
\Leftrightarrow\; &\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{1:H-1}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}) = \tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i, \mathbf{m}_{-i}^{2:H}),\ \forall \mathbf{m}_i.
\end{aligned} \tag{32}$$

With Equation (32) and the condition of equal choice probability, it is straightforward to show

$$\begin{aligned}
&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1(2:H)}) \\
=&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2:H)}) \\
=&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2:H)}) \\
=&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{2(2:H)}).
\end{aligned} \tag{33}$$

The second and forth lines are the results of Equation (32) and the second equality is due to the condition of equal choice probability. This condition could further implies that Equation (33) could be re-arranged as:

$$\begin{aligned}
&\tilde{\mathbb{F}}_i^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1:H-1)}) \\
&\cdot[\tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{1(2:H)}) - \tilde{\mathbb{p}}_{-i}^{-1}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1:H-1)}) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^{2(2:H)})] \\
=&\mathbf{0}.
\end{aligned} \tag{34}$$

Note that $\tilde{\mathbb{F}}_i^{-1}(\cdot)$ is a matrix with full rank due to the full rank condition of $\mathbf{\Delta}_i(\cdot)$. Therefore, Equation (34) consists of $min\big(K_i, \prod_{j\neq i}(K_j+1)-1\big)$ linear restrictions imposed on the matrix in the bracket []. It then suggests that the rank of this matrix is at most $\prod_{j\neq i}(K_j+1)-1-min\big(K_i,\prod_{j\neq i}(K_j+1)-1\big)$. This completes the proof. $\square$

As compared to Proposition 1 that focuses on 2×2 games, Proposition 1' requires two slightly stronger conditions. These two conditions would still hold generically when the control variable $\mathbf{m}$ has sufficient variation. Specifically, the first condition states that the equal choice probability property is satisfied for $\prod_{j\neq i}(K_j+1)+1$ pairs of realizations, rather than just three pairs in the case of 2×2 games. These additional pairs are essential for our results due to extra dimensions of the action profiles in a general multinomial choice game. The second condition implicitly assumes the matrix $\tilde{\mathbb{p}}_{-i}(\cdot)$ is invertible. This invertibility obviously requires $\mathbf{m}_{-i}$ to affect $\mathbf{p}_{-i}(\cdot)$, as in the case of 2×2 games. However, it further requires the effect to cause "linearly independent" variation of $p_{-i}(\mathbf{a}_{-i}|\mathbf{m})$ so that the choice probability of any action profile cannot be written as a linear combination of the probabilities of other profiles. When $\mathbf{m}_{-i}$ has enough dimensions and sufficient variation, the invertibility condition will be satisfied with probability 1. This is because in the space of all possible matrices, the set of invertible matrices has a full measure.

The two conditions in Proposition 1' could be easily generated through an experimental design. Consider the utility function such that $\pi_i(\mathbf{m}_i,\mathbf{a}) = u_i[m_i(\mathbf{a})]$. Suppose that the analyst designs an experiment that exogenously varies $m_i(\mathbf{a})$, independent of the action profile $\mathbf{a}$ and player role i. Such a design ensures that the matrix $\frac{\partial \boldsymbol{E}\tilde{\pi}_i(\cdot)}{\partial \mathbf{m}_i'}$ has a full rank for almost every realization of $\mathbf{m}_i$. This full rank property is a sufficient condition for the equal choice probability property in Proposition 1'. In addition, the independent variation of $\mathbf{m}_{-i}(a_i,\mathbf{a}_{-i}=k)$ could lead to an independent variation of $p_{-i}(\mathbf{a}_{-i}=k|\mathbf{m})$. This linear independence directly implies the invertibility of the matrix $\tilde{\mathbb{p}}_{-i}(\cdot)$.

Unlike Proposition 1 in the case of 2×2 games that provides an equality restriction, Proposition 1' derives a rank restriction as the testable implication of QRE. In particular, Equation (28) imposes an upper bound on a matrix of player $-i$'s choice probability function. It implies that such a matrix cannot be full rank. Importantly, this restriction has the power to reject incorrect null hypothesis, since the set of rank deficient matrices has a zero measure. Moreover, to test the implication by Equation (28), the analyst has to do inference on the rank of a matrix. This technique has been well developed in the econometrics literature (Robin and Smith, 2000; Kleibergen and Paap, 2006; Camba-Mendez and Kapetanios, 2009). Finally, Proposition 1' is the result for a general game and Proposition 1 can be seen as a special case when there are two players and two actions.[21]

In games with more than two players and/or actions, the identification of the utility and the distribution functions requires two slightly stronger assumptions. The first one imposes a so-called *rank ordering property* and is summarized by Assumption 3'. It modifies Assumption 3 in 2×2 games to a general multinomial choice game.

Assumption 3'. *(a) There exists a realization* $\mathbf{m}_i = \mathbf{m}_i^1$*, such that* $\left|\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0})\right| = \left|\pi_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0}) - \pi_i(\mathbf{m}_i^1, a_i = 0, \mathbf{a}_{-i} = \mathbf{0})\right| = 1$.

(b) For any two actions, denoted by k *and* k'*, we have* $E\pi_i[\mathbf{m}_i, a_i = k, \mathbf{p}_{-i}(\mathbf{m})] > E\pi_i[\mathbf{m}_i, a_i = k', \mathbf{p}_{-i}(\mathbf{m})]$ *if and only if* $p_i(a_i = k|\mathbf{m}) > p_i(a_i = k'|\mathbf{m})$.

Assumption 3'(a) is identical to Assumption 3(a). In addition, Assumption 3'(b) is known as the rank ordering property. It was first introduced by Manski (1975) and subsequently exploited by Goeree et al. (2005) and Goeree et al. (2019) in the QRE literature. The rank ordering property only imposes one weak restriction on the distribution function and choice probability. In particular, action k is chosen more frequently than action k' if and only if action k has a higher expected utility. This property is satisfied under the Logit specification and many other distribution functions. One simple case is that $\varepsilon_i(a_i)$ follows an identical non-parametric distribution function and is independent across actions. Moreover, Assumption 3' could also be satisfied when the perturbation

[21] In particular, in case of 2×2 games, the right hand side of Equation (28) is zero. It suggests that the matrix in the left hand side must be the one whose every element is zero. In 2×2 games, this matrix essentially reduces to the expression by Equation (7). Consequently, Equation (7) in Proposition 1 can be seen as a special case of Equation (28) when there are two players and two actions.

is correlated across actions. Consider that ϵ_i follows a multivariate normal distribution with the restrictions $Var(\varepsilon_i(k)) = \sigma^2\ \forall k$ and $Cov(\varepsilon_i(k), \varepsilon_i(k')) = \rho\sigma^2$. This error structure with fixed variance and covariance across actions would imply the rank ordering property. Further, Goeree et al. (2005) consider another class of distribution functions that satisfy the *exchangeability*. Specifically, it requires the distribution function to be fixed for any perturbation of ϵ_i. As shown by Goeree et al. (2005), this exchangeability is a sufficient condition for the rank ordering property. At last, when each player has two actions, Assumption 3'(b) is equivalent to the zero median restriction and reduces to Assumption 3(b). Similar as in the main text, the rank ordering property would be redundant and is testable in experimental settings that specify $\pi_i(\mathbf{m}_i, \mathbf{a}) = u_i[m_i(\mathbf{a})]$.

The second condition to establish the identification results imposes a weak restriction on the choice probability function. Definition 2 introduces two terminologies that facilitate the expression of the restriction.

Definition 2. *(a) For each player i, a pair of actions k and k′ is* ***directly connected*** *at* $\mathbf{m}_i = \mathbf{m}_i^1$ *if there exists a realization of* $\mathbf{m}_{-i}$*, say* $\mathbf{m}_{-i}^1 \in int[Supp(\mathbf{m}_{-i})]$*, such that* $p_i(a_i = k|\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p_i(a_i = k'|\mathbf{m}_i^1, \mathbf{m}_{-i}^1)$.

(b) For each player i, a pair of actions k and k′ is ***indirectly connected*** *at* $\mathbf{m}_i = \mathbf{m}_i^1$ *if these two actions are not directly connected. However, there exists a sequence of actions* $\{k, l, l', l'', \cdots, k'\}$ *where each pair of adjacent actions in this sequence is directly connected.*

We define two actions to be connected if they are chosen with equal probability under some realizations of $\mathbf{m}$. With the rank ordering property, two actions are connected if they have the same expected utility for some $\mathbf{m}$. This condition is straightforward to generate in a lab experiment. In particular, consider the utility function $\pi_i(\mathbf{m}_i, \mathbf{a}) = u_i[m_i(\mathbf{a})]$. The analyst could carefully choose the monetary payoff so that the domain of the variable $m_i(a_i = k, \mathbf{a}_{-i})$ has a non-empty intersection with the domain of $m_i(a_i = k', \mathbf{a}'_{-i})$. Consequently, sufficient variation of $\mathbf{m}_{-i}$ would ensure that the expected utilities of these two actions equalize at some realization $\mathbf{m}_{-i} = \mathbf{m}^1_{-i}$. The identification results in this paper require each pair of player i's actions to be either directly connected or indirectly connected. With this restriction, Proposition 2' establishes the identification of the inverted choice probability function $\mathbf{F}_i^{-1}(\cdot)$. Note that in a binary choice game, the condition of connection reduces to the restriction that player i's choice probability function $p_i(\mathbf{m}_i, \mathbf{m}_{-i})$ crosses the point 1/2. The latter restriction is imposed in Proposition 2 that focuses on 2×2 games.

Proposition 2'. *Under Assumptions 1, 2', and 3', and suppose that QRE restrictions are satisfied whenever $\mathbf{m}_i = \mathbf{m}_i^1$ regardless of the realization of $\mathbf{m}_{-i}$. Furthermore, for any two actions $k \neq k'$ of player i, suppose that they are either directly connected or indirectly connected at $\mathbf{m}_i = \mathbf{m}_i^1$. Then the inverted choice probability function $\mathbf{F}_i^{-1}(\mathbf{p})$ is point identified $\forall \mathbf{p} \in \mathcal{P}_i(\mathbf{m}_i^1)$.*

Proof. Without loss of generality, suppose that action 0 and action 1 are directly connected at $\mathbf{m}_i = \mathbf{m}_i^1$. First consider the case of a two-player game where player j has binary choice; for instance, $\prod_{j \neq i}(K_j + 1) = 2$. Following the proof of Proposition 2, it directly identifies $\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i})$ $\forall \mathbf{a}_{-i}$. Next, consider a general game (with potentially more than two players) where $\prod_{j \neq i}(K_j + 1) > 2$, the condition that actions 0 and 1 are connected implies the following: Generically, there exist infinite realizations of $\mathbf{m}_{-i}$ that equalize these two actions' expected utilities. To see this point, consider the condition of equal expected utility that $E\pi_i[\mathbf{m}_i^1, a_i = 1, \mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})] = E\pi_i[\mathbf{m}_i^1, a_i = 0, \mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})]$. It is an equation but with multiple unknowns (i.e., $\mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})$). With sufficient variation of $\mathbf{m}_{-i}$, there are infinite solutions (i.e., the value of vector $\mathbf{p}_{-i}(\cdot)$) of this equation. Let us consider $H = \prod_{j \neq i}(K_j + 1) - 1$ such solutions, denoted by $\mathbf{m}^1_{-i}$ to

$\mathbf{m}_{-i}^{H}$. It then implies the following equation system:

$$\begin{aligned}
&\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0}) \\
&+ \sum_{\mathbf{a}_{-i} \neq \mathbf{0}} [\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i}) - \tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0})] \cdot p_{-i}(\mathbf{a}_{-i} | \mathbf{m}_i^1, \mathbf{m}_{-i}^h) \\
&= 0, \quad \forall h \leq H.
\end{aligned} \tag{35}$$

Equation (35) can be seen as a linear system that consists of H restrictions and H unknowns; for instance, the unknowns are $\tilde{\pi}_i(\cdot, \mathbf{a}_{-i}) - \tilde{\pi}_i(\cdot, \mathbf{a}_{-i} = 0)$ $\forall \mathbf{a}_{-i} \neq \mathbf{0}$. Consequently, each of these terms is identified as a linear transformation of $\pi_i(\cdot, a_i = 1, \mathbf{a}_{-i} = \mathbf{0})$. Next, consider any value of $\mathbf{m}_{-i}$ such that $p_i(a_i = 1 | \mathbf{m}_i^1, \mathbf{m}_{-i}) > (<) p_i(a_i = 0 | \mathbf{m}_i^1, \mathbf{m}_{-i})$. The rank ordering property then implies that the difference of expected utility is positive (negative); for instance, $\tau[\mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})] \cdot \tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0}) > (<) 0$. Note that $\tau(\cdot)$ represents the linear transformation that has been identified by Equation (35). Consequently, this relationship identifies the sign of $\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0})$. Further, given the scale normalization by Assumption 3'(a), the absolute value of this term is normalized to 1. As a result, the value of $\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i} = \mathbf{0})$ is identified. Finally, by Equation (35), it further identifies $\tilde{\pi}_i(\mathbf{m}_i^1, a_i = 1, \mathbf{a}_{-i})$ $\forall \mathbf{a}_{-i}$.

The condition of Proposition 2' imposes that every action is either directly or indirectly connected to action 0. Therefore, by a similar argument as above, we can establish the identification of $\tilde{\pi}_i(\mathbf{m}_i^1, \mathbf{a})$ for every action profile $\mathbf{a}$. Given these terms, consider the evaluation of Equation (27) at the realization $\mathbf{m}_i = \mathbf{m}_i^1$, as represented by the following:

$$\tilde{\boldsymbol{\pi}}_i(\mathbf{m}_i^1) + \boldsymbol{\Delta}_i(\mathbf{m}_i^1) \cdot \mathbf{p}_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}) = \mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m}_i^1, \mathbf{m}_{-i})].$$

In the above equation, every term on the left hand side is either identified or observed. Moreover, the variation of $\mathbf{m}_{-i}$ could exogenously vary $\mathbf{p}_{-i}(\cdot)$. This variation then identifies $\mathbf{F}_{-i}^{-1}(\mathbf{p})$ $\forall \mathbf{p} \in \mathcal{P}_i(\mathbf{m}_i^1)$. It completes the proof. $\square$

Due to the bijectivity result by Hotz and Miller (1993) and Norets and Takahashi (2013), the identification of the inverse function $\mathbf{F}_i^{-1}(\cdot)$ – as established in Proposition 2' – directly implies the identification of the distribution function of ϵ_i. Moreover, similar as the argument in the main text, when the QRE restriction is imposed on sufficiently many but a finite number of realizations of $\mathbf{m}_i$, function $\mathbf{F}_i^{-1}[\mathbf{p}_i(\mathbf{m})]$ could be point identified for almost the entire image of player i's choice probability function $\mathbf{p}_i(\mathbf{m})$.

The last result of this generalization establishes the identification of the utility function $\pi_i(\cdot)$, as stated in Proposition 3'.

Proposition 3'. *Suppose that the conditions met in Proposition 3 hold so that $\mathbf{F}_i^{-1}(\cdot)$ is identified. Moreover, consider $H = \prod_{j \neq i}(K_j + 1)$ realizations of $\mathbf{m}_{-i}$, denoted by $\mathbf{m}_{-i}^1$ to $\mathbf{m}_{-i}^H$. Suppose that QRE restrictions are satisfied whenever $\mathbf{m}_{-i} = \mathbf{m}_{-i}^h$ $\forall h \leq H$, regardless of the realization of $\mathbf{m}_i$. These conditions imply that the difference of utility function $\tilde{\pi}_i(\mathbf{m}_i, a_i, \mathbf{a}_{-i}) = \pi_i(\mathbf{m}_i, a_i, \mathbf{a}_{-i}) - \pi_i(\mathbf{m}_i, a_i = 0, \mathbf{a}_{-i})$ is point identified $\forall \mathbf{m}_i, a_i, \mathbf{a}_{-i}$.*

Proof. Consider an arbitrary action $a_i = k$ of player i and all the realizations $\mathbf{m}_{-i} = \mathbf{m}_{-i}^h$ such that QRE restrictions hold. Evaluating Equation (27) at this action and these realizations would imply the following relationship:

$$\sum_{\mathbf{a}_{-i}} \tilde{\pi}_i(\mathbf{m}_i, a_i = k, \mathbf{a}_{-i}) \cdot p_{-i}(\mathbf{a}_{-i} | \mathbf{m}_i, \mathbf{m}_{-i}^h) = F_{i,k}^{-1}[\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}_{-i}^h)], \ \forall h \leq H. \tag{36}$$

Equation (36) is a system that consists of H linear restrictions and H unknowns (i.e., $\tilde{\pi}_i(\cdot)$). The rank condition is satisfied. Therefore, $\tilde{\pi}_i(\mathbf{m}_i, a_i, \mathbf{a}_{-i})$ is point identified $\forall \mathbf{m}_i, a_i, \mathbf{a}_{-i}$. It completes the proof. □

Relaxing the Invariance Assumption

This section relaxes the invariance assumption and derives a testable implication of QRE under a weaker restriction. In particular, we allow the distribution of player i's mistake to depend on their own utility, but restrict it to be independent of other players' utilities.

For generality, we consider the general multinomial choice game described in the above section. Naturally, it nests the 2×2 game – as in the main text – as a special case. The following Assumption 2'' presents a weaker invariance restriction.

Assumption 2''. *$\Gamma_i(\cdot)$ is independent of $\mathbf{m}_{-i}$, but could depend on $\mathbf{m}_i$. For instance, $\Gamma_i(\epsilon_i|\mathbf{m}_i, \mathbf{m}_{-i}) = \Gamma_i(\epsilon_i|\mathbf{m}_i)$ $\forall 1 \leq i \leq N$.*

Under Assumption 2'', Proposition 4 derives a testable implication of QRE.

Proposition 4. *Under Assumptions 1 and 2'' and consider any $H = \prod_{j \neq i}(K_j + 1)$ realizations of $\mathbf{m}_{-i}$ — denoted by $\mathbf{m}^h_{-i}$ $\forall h \leq H$ — such that $\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^h_{-i}) = \mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^{h'}_{-i})$ $\forall h \neq h'$. Given these realizations, QRE implies the following testable restriction:*

$$Rank[\tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}^{1:H}_{-i})] \leq \prod_{j \neq i}(K_j + 1) - 1 - min(K_i, \prod_{j \neq i}(K_j + 1) - 1). \tag{37}$$

Note that $\tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}^{1:H}_{-i})$ is a $\left(\prod_{j \neq i}(K_j + 1) - 1\right) \times \left(\prod_{j \neq i}(K_j + 1) - 1\right)$ matrix.

Proof. Recall Equation (30), it directly implies the following:

$$\mathbf{\Delta}_i(\mathbf{m}_i) \cdot \tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}^{1:H}_{-i}) = \tilde{\mathbb{F}}^{-1}_i(\mathbf{m}_i, \mathbf{m}^{1:H}_{-i}) = \mathbf{0}. \tag{38}$$

The second equality of Equation (38) is due to the condition of equal choice probability that $\mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^h_{-i}) = \mathbf{p}_i(\mathbf{m}_i, \mathbf{m}^{h'}_{-i})$. Since we focus on the generic case that $\mathbf{\Delta}_i(\cdot)$ has a full rank, Equation (38) then consists of $min(K_i, \prod_{j \neq i}(K_j + 1) - 1)$ linearly independent restrictions on the $[\prod_{j \neq i}(K_j + 1) - 1] \times [\prod_{j \neq i}(K_j + 1) - 1]$ matrix $\tilde{\mathbb{p}}_{-i}(\mathbf{m}_i, \mathbf{m}^{1:H}_{-i})$. Consequently, the rank of this matrix is at most $\prod_{j \neq i}(K_j + 1) - 1 - min(K_i, \prod_{j \neq i}(K_j + 1) - 1)$. It completes the proof. □

Compared with the results under the invariance assumption, the testable implication under the relaxed restriction by Assumption 2'' requires a similar but slightly stronger condition of equal choice probability. In more detail, Propositions 1 and 1' restrict player

i's choice probability to be fixed across some games with variations of both $\mathbf{m}_i$ and $\mathbf{m}_{-i}$. In contrast, Proposition 4 requires this equal choice property to hold across games with variation of only $\mathbf{m}_{-i}$, but under a fixed value of own control variable $\mathbf{m}_i$. This slightly stronger condition *with fixed* $\mathbf{m}_i$ is generally satisfied in our framework. However, there exist a few exceptions. An important violation is in the case of 2×2 game where the utility structure satisfies the following two restrictions: (i) $\mathbf{m}_i$ is a single variable represented by m_i, and (ii) player i's choice probability function $p_i(m_i, m_{-i})$ is strictly monotone in both m_i and m_{-i}. Note that the matching pennies game as shown by Table 1 satisfies the above two conditions. As shown by Figure 1, the equal choice probability property with fixed m_i can never hold in this game. In contrast, this condition can be always satisfied in any game as long as $p_i(\cdot)$ is not monotone in m_{-i} or the control variables $\mathbf{m}_{-i}$ have sufficient dimensions.[22] To see this point, consider again the matching pennies game but suppose there are two variables $\mathbf{m}_{-i} = (m_{-i,0}, m_{-i,1})$ that could affect player $-i$'s utility. Further, assume that $m_{-i,0}$ strictly raises player $-i$'s utility of action 0 while $m_{-i,1}$ increases the utility of action 1. The structure of matching pennies game then implies $p_i(\mathbf{m}_i, m_{-i,0}, m_{-i,1})$ is strictly decreasing in $m_{-i,0}$ and strictly increasing in $m_{-i,1}$. Consequently, for any fixed value of $\mathbf{m}_i$, the analyst could always find infinite pairs of $(m_{i,0}, m_{-i,1})$ that equalize player i's choice probability. In other words, the condition of equal choice probability with fixed $\mathbf{m}_i$ can be always satisfied in this game.

Explanation of the Normalizations

This section describes in details about our normalization in Assumption 3. It explains why these normalizations are innocuous and their relationship to our identification results. First consider the transformation such that $\hat{\pi}_i(\cdot) = \alpha + \beta \pi_i(\cdot)$, $\hat{\varepsilon}_i(\cdot) = \beta \varepsilon_i(\cdot)$ where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Any such affine transformation preserves the same preference and

[22]There are two sources to increase the dimension of $\mathbf{m}_{-i}$. The first one is provided by each player i to have more variables in their own $\mathbf{m}_i$. The second source is by increasing the number of players.

predicts identical choice. Therefore, α and β have to be normalized or dealt with. In the main text, we deal with α by identifying the difference of utility function $\tilde{\pi}_i(\cdot)$, as established in Proposition 3. This difference would cancel out α.

Normalizing β is usually done by setting the *scale* (or absolute value) of the utility $\pi_i(\mathbf{m}_i^1, \mathbf{a})$ under *only one* realization $\mathbf{m}_i^1$ and under *only one* action profile $\mathbf{a}$ to be 1. In contrast, the utility of any other profile under any other realization of $\mathbf{m}_i$ is unrestricted. In Assumption 3(a), we impose an equivalent normalization that sets the scale of the utility difference $|\pi_i(\mathbf{m}_i^1, a_i = 1, a_{-i} = 0) - \pi_i(\mathbf{m}_i^1, a_i = 0, a_{-i} = 0)|$ to be 1. We choose this normalization as it is simple to derive our identification results.

To better understand the above scale normalization, consider the matching pennies game in Table 1 and the specification $\pi_i(\mathbf{m}_i, \mathbf{a}) = u_i[m_i(\mathbf{a})]$. For an arbitrary value of m_1^1, Assumption 3(a) normalizes $|u_1(8) - u_1(m_1^1)| = 1$. Together with a *location normalization* that $u_1(8) = 0$ (i.e., the utility of the minimum monetary reward is zero), it further implies that $u_1(m_1^1) = 1$. Therefore, our analysis imposes normalization on two points of the utility function $u_1(m)$ (i.e., $m = 8, m_1^1$) while the value of $u_1(m)$ is unrestricted for any other value of m. These restrictions on two points of the utility function are equivalent to imposing restrictions on both α and β in the affine transformation $\hat{u}_i(\cdot) = \alpha + \beta u_i(\cdot)$. Since any such transformation preserves the same preference, our normalization is innocuous.

To see the normalization in Assumption 3(b), consider another transformation such that $\hat{\tilde{\pi}}_i(\cdot) = \tilde{\pi}_i(\cdot) + \alpha$ and $\hat{\tilde{\varepsilon}}_i = \tilde{\varepsilon}_i - \alpha$. This additive transformation is applied to the *utility difference* and *error difference*. It is *distinct* from the affine transformation – as described in the above paragraphs – that applies to the *utility of each action profile*. As shown by Equation (1), this additive transformation on the difference of utility and error also preserves the same choice. Consequently, a location normalization is required and is imposed by Assumption 3(b) that normalizes the median of $\bar{\varepsilon}_i$ to be zero. Note that Assumption 3(b) is equivalent to the following condition: Suppose that two actions have the same expected utility, then these two actions must be chosen with the same probability. This condition is satisfied for Logit and Probit specifications. It is exploited in almost every empirical application of QRE. Importantly, it is also the key restriction in the regular QRE (Goeree et al., 2005) and the rank-dependent choice equilibrium (Goeree et al., 2019).

At last, Assumption 3(b) is only required for the general specification of the utility function $\pi_i(\mathbf{m}_i, \mathbf{a})$. When the utility is defined only on the space of received monetary reward (i.e., $u_i[m_i(\mathbf{a})]$), the median of $\tilde{\varepsilon}_i$ can be identified. Therefore, Assumption 3(b) is redundant and testable under this scenario. To see this point, recall that $m_i(\mathbf{a})$ could be designed to have an independent variation across each action profile. Therefore, there exists a realization $\mathbf{m}_i^1$ such that $m_i^1(a_i = 0, a_{-i} = 0) = m_i^1(a_i = 1, a_{-i} = 1)$ and $m_i^1(a_i = 1, a_{-i} = 0) = m_i^1(a_i = 0, a_{-i} = 1)$. This payoff structure could appear in matching pennies games and coordination games. Under this realization $\mathbf{m}_i^1$, Equation (20) turns to:

$$F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i})] = \tilde{u}_i^1 \cdot [1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})], \tag{39}$$

where $\tilde{u}_i^1 = u_i[m_i^1(a_i = 1, a_{-i} = 0)] - u_i[m_i^1(a_i = 0, a_{-i} = 0)] = u_i[m_i^1(a_i = 0, a_{-i} = 1)] - u_i[m_i^1(a_i = 1, a_{-i} = 1)]$. Under Assumption 3(a) that normalizes $|\tilde{u}_i^1| = 1$, Equation (39) reduces to:

$$F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i})] = sign(\tilde{u}_i^1) \cdot [1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})]. \tag{40}$$

Consequently, variation of $\mathbf{m}_{-i}$ could identify the inverse of distribution function $F_i^{-1}(\cdot)$. This identification is achieved without imposing $Median(\tilde{\varepsilon}_i) = 0$. Therefore, the median of the error distribution can be identified.