Non-Parametric Identification and Testing of Quantal Response Equilibrium

Online Appendix: Additional Testable Implications

Johannes Hoelzemann   Ryan Webb   Erhao Xie

March 14, 2024

In Proposition 5, we first show that our over-identification test includes all the testable implications derived by Xie (2022).

Proposition 5. *Suppose that Assumptions 1 to 4 hold. If Equation* (7) *is satisfied, then the QRE restrictions in Equation* (8) *hold for any three pairs of games such that* $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(l)}) = (\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(l)})$ *for* $l = 1,2,3$ *and* $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) \neq p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})$.

Proof. Recall the definition of Equation (6). Consider two distinct realizations of $\mathbf{m}_{-i}$, denoted as $\mathbf{m}_{-i}^{1(1)}$ and $\mathbf{m}_{-i}^{1(2)}$. When we individually substitute these realizations into Equation (6) and subtract one from the other, we obtain the following equation:

$$
\begin{aligned}
&\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})|\mathbf{m}_i^1] - \hat{F}_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1] \\
=&[\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot [p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)}) - p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})]. \qquad (28)
\end{aligned}
$$

By a similar argument, for realizations $\mathbf{m}_{-i}^{1(1)}$ and $\mathbf{m}_{-i}^{1(3)}$, we could derive the following:

$$
\begin{aligned}
&\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(3)})|\mathbf{m}_i^1] - \hat{F}_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1] \\
=&[\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot [p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(3)}) - p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})]. \qquad (29)
\end{aligned}
$$

Dividing Equation (29) by Equation (28) would imply the following ratio:

$$\frac{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(3)})|\mathbf{m}_i^1]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1]}{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(2)})|\mathbf{m}_i^1]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1]}=\frac{p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(3)})-p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})}{p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(2)})-p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})}. \tag{30}$$

Repeating the above steps for another realization $\mathbf{m}_i^2$, one could derive a similar equation:

$$\frac{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(3)})|\mathbf{m}_i^2]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})|\mathbf{m}_i^2]}{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(2)})|\mathbf{m}_i^2]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})|\mathbf{m}_i^2]}=\frac{p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(3)})-p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})}{p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(2)})-p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})}. \tag{31}$$

Let us consider any three pairs of games that satisfy the condition of equal choice probability; for instance $p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(l)})=p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(l)})$ for $l=1,2,3$. A combination of Equations (30) and (31) would imply the following relationship:

$$\begin{aligned}\frac{p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(3)})-p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})}{p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(2)})-p_{-i}(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})}&=\frac{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(3)})|\mathbf{m}_i^1]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1]}{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(2)})|\mathbf{m}_i^1]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(1)})|\mathbf{m}_i^1]}\\&=\frac{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(3)})|\mathbf{m}_i^2]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})|\mathbf{m}_i^2]}{\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(2)})|\mathbf{m}_i^2]-\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})|\mathbf{m}_i^2]}\\&=\frac{p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(3)})-p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})}{p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(2)})-p_{-i}(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(1)})}.\end{aligned} \tag{32}$$

The first and third lines in Equation (32) are direct results of Equations (30) and (31). The second line follows the equal choice probability condition and Equation (7) so that $\hat{F}_i^{-1}[p_i(\mathbf{m}_i^1,\mathbf{m}_{-i}^{1(l)})|\mathbf{m}_i^1]=\hat{F}_i^{-1}[p_i(\mathbf{m}_i^2,\mathbf{m}_{-i}^{2(l)})|\mathbf{m}_i^2]$ for $l=1,2,3$. This completes the proof.

□

Following Proposition 5, if Xie (2022)'s testable implication is violated, our over-identification test in Proposition 2 would also reject QRE. Importantly, the reverse is not true since our test includes more restrictions than Xie (2022) and has higher statistical power. Specifically, the structure of monetary payoffs often implies additional restrictions on players' utilities across games or action profiles. As shown in Subsection 3.3, each player's utility function is non-parametrically identified. Therefore, these utility restrictions become testable implications of QRE in addition to the ones derived by Xie (2022). To better describe these results, we extend Assumption 6 to include another structural property of the matching pennies game presented in Table 1. The property, indexed as Assumption 6(c), preserves player i's payoffs for one of the other player's actions and varies player i's payoffs when player $-i$ chooses the other action.

Assumption 6. *(c) For each player i, there exist two realizations of* $\mathbf{m}_i$*—denoted as* $\mathbf{m}_i^1$ *and* $\mathbf{m}_i^2$*—such that* $m_i^1(a_i, a_{-i}) = m_i^2(a_i, a_{-i})\ \forall a_i$ *and for some* a_{-i}*.*

The strict monotonicity of the utility function and each condition in Assumption 6 implies different testable implications of QRE. Proposition 6 shows that these implications are included in our over-identification test.

Proposition 6. *Suppose that Assumptions 1 to 4 hold, then Equation* (7) *implies the following testable restrictions of QRE:*

(a) $\forall \mathbf{m}_i \in \mathcal{M}_i$*:*

$$Sign\{\frac{(-1)^{a_{-i}} p_{-i}(1-a_{-i}|\mathbf{m}_i, \mathbf{m}_{-i}^2) F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}_{-i}^1)] + (-1)^{1-a_{-i}} p_{-i}(1-a_{-i}|\mathbf{m}_i, \mathbf{m}_{-i}^1) F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}_{-i}^2)]}{p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^1) - p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^2)}\}$$

$$= Sign[m_i(a_i = 0, a_{-i}) - m_i(a_i = 1, a_{-i})], \ \ \forall \mathbf{m}_{-i}^1, \mathbf{m}_{-i}^2 \in \mathcal{M}_{-i} \text{ and } \forall a_{-i}. \tag{33}$$

(b) $\forall \mathbf{m}_i^1$ *that satisfies Assumption 6(a):*

$$\frac{1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1)}{1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^2)} = \frac{F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1)]}{F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^2)]}, \ \forall \mathbf{m}_{-i}^1, \mathbf{m}_{-i}^2 \in \mathcal{M}_{-i}. \tag{34}$$

(c) For each pair of $\mathbf{m}_i^1$ *and* $\mathbf{m}_i^2$ *that satisfies Assumption 6(b):*

$$\frac{(-1)^{a_{-i}} p_{-i}(1-a_{-i}|\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)}) F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})] + (-1)^{1-a_{-i}} p_{-i}(1-a_{-i}|\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})]}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) - p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})} =$$

$$-\frac{(-1)^{a'_{-i}} p_{-i}(1-a'_{-i}|\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)}) F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})] + (-1)^{1-a'_{-i}} p_{-i}(1-a'_{-i}|\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}) F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})]}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}) - p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})},$$

$$\forall \mathbf{m}_{-i}^{1(1)}, \mathbf{m}_{-i}^{1(2)}, \mathbf{m}_{-i}^{2(1)}, \mathbf{m}_{-i}^{2(2)} \in \mathcal{M}_{-i}. \tag{35}$$

(d) Consider each pair of $\mathbf{m}_i^1$ and $\mathbf{m}_i^2$ that satisfies both Assumption 6(c) and the condition that $\mathcal{P}_i(\mathbf{m}_i^1) \cap \mathcal{P}_i(\mathbf{m}_i^2)$ includes an interval. Then for any two pairs of games such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(l)}) = p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(l)})$ for $l = 1, 2$, we have:

$$\frac{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})} = \frac{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})}. \tag{36}$$

Proof. For any $\mathbf{m}_i$, consider two realizations denoted as $\mathbf{m}^1_{-i}$ and $\mathbf{m}^2_{-i}$. Evaluating the definition of $\hat{F}_i^{-1}(p|\mathbf{m}_i)$ by Equation (6) at these two realizations leads to the following system:

$$\begin{aligned}
F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^1_{-i})] &= \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i}) \\
F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^2_{-i})] &= \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i, \mathbf{m}^2_{-i}).
\end{aligned} \tag{37}$$

In the left hand side of this system, we replace $\hat{F}_i^{-1}(p|\mathbf{m}_i)$ by $F_i^{-1}(p)$. This follows the implication by Equation (7) such that $\hat{F}_i^{-1}(p|\mathbf{m}_i) = F_i^{-1}(p)\ \forall \mathbf{m}_i$. This linear system by Equation (37) identifies utility difference $\tilde{\pi}_i(\cdot)$ as the following expression:

$$\begin{aligned}
&\tilde{\pi}_i(\mathbf{m}_i, a_{-i}) \qquad (38)\\
=&\frac{(-1)^{a_{-i}} p_{-i}(1 - a_{-i}|\mathbf{m}_i, \mathbf{m}^2_{-i}) F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^1_{-i})] + (-1)^{1-a_{-i}} p_{-i}(1 - a_{-i}|\mathbf{m}_i, \mathbf{m}^1_{-i}) F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^2_{-i})]}{p_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i}) - p_{-i}(\mathbf{m}_i, \mathbf{m}^2_{-i})}.
\end{aligned}$$

Note that there always exist $\mathbf{m}^1_{-i}$ and $\mathbf{m}^2_{-i}$ such that the denominator in the second line is non-zero. This is because $p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i})$ varies with $\mathbf{m}_{-i}$.

The property of the utility function and the structure of monetary payoffs impose restrictions on $\tilde{\pi}_i(\cdot)$. It is these restrictions that lead to Proposition 6. Specifically, the strict increasing property of $u_i(m)$ implies that $\tilde{\pi}_i(\mathbf{m}_i, a_{-i})$ and $[m_i(a_i = 0, a_{-i}) - m_i(a_i = 1, a_{-i})]$ have the same sign. It leads to Proposition 6(a). Assumption 6(a) suggests that

$\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i}) = -\tilde{\pi}_i(\mathbf{m}_i^1, 1-a_{-i})$. It leads to Proposition 6(b). Assumption 6(c) restricts $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i}) = -\tilde{\pi}_i(\mathbf{m}_i^2, a'_{-i})$ and implies Proposition 6(c).

To prove Proposition 6(d), assume that Assumption 6(c) holds for the action $a_{-i} = 1$. The proof for the case that $a_{-i} = 0$ follows a similar argument and is suppressed. By transforming the system by Equation (37), we obtain the following relationship:

$$\frac{p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^1)}{p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}^2)} = \frac{F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}_{-i}^1)] - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1)}{F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}_{-i}^2)] - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1)}. \tag{39}$$

Now consider two pairs of games such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(l)}) = p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(l)})$ for $l = 1, 2$, we then obtain Proposition 6(d) through the following steps:

$$\begin{aligned}\frac{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})} &= \frac{F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})] - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)}{F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})] - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)} \\ &= \frac{F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})] - \tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1)}{F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})] - \tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1)} \\ &= \frac{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})}. \end{aligned} \tag{40}$$

The first and third lines follow directly from Equation (39). The second line is due to the equal choice probability condition that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(l)}) = p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(l)})$ and the implication of Assumption 6(c) such that $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) = \tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1)$. This completes the proof. $\square$

Given Proposition 1 that identifies $F_i^{-1}(\cdot)$, all equations in Proposition 6 are then restrictions on known functions of players' choice probabilities and are therefore testable.

In Proposition 6, restriction (a) exploits only the strict monotonicity of $u_i(m)$ and applies to any payoff structure and any type of games. In contrast, restrictions (b) to (d) focus on matching pennies games. In addition, restrictions (a) to (c) do not require the equal choice probability condition. Restriction (d) requires this condition but only for two pairs of games as opposed to the three pairs in Xie (2022). Therefore, all restrictions in Proposition 6 are additional testable implications in our over-identification test, but they are excluded from Xie (2022).

Even though Proposition 6(b) to (d) focuses on matching pennies games, other types of games have their own monetary payoffs structure. These structural properties can be exploited to derive additional testable restrictions of QRE. For instance, consider the coordination game illustrated in Table 2. Assumption 6(b) holds when the analyst considers two values $m_i^1 = 0$ and $m_i^2 = 15$. Therefore, Proposition 6(c) applies. Moreover, in this coordination game, the payoff of $a_i = 0$ does not depend on the other player's action. It implies that $\tilde{\pi}_i(m_i, a_{-i} = 0) - \tilde{\pi}_i(m_i, a_{-i} = 1) = u_i(15) - u_i(0)$, which is independent of m_i. Consequently, the following is a natural testable implication of QRE:

$$\frac{F_i^{-1}[p_i(m_i, m_{-i}^1)] - F_i^{-1}[p_i(m_i, m_{-i}^2)]}{p_{-i}(m_i, m_{-i}^1) - p_{-i}(m_i, m_{-i}^2)} \text{ is independent of } m_i,\ \forall m_{-i}^1, m_{-i}^2. \tag{41}$$

In another section of this online appendix (i.e., Section "Generalizations and Extensions"), we consider an experiment that varies at least two action profiles' payoffs without further restrictions on the payoff structures. For instance, it does not require Assumption 6. This general structure includes common types of games as special cases. In that section, we demonstrate that there are additional testable restrictions of QRE.

Even though experimental data typically provides additional structure to test QRE, the concrete form of the restrictions depends on the structure of $\mathcal{M}$ and is therefore application specific. While it can be cumbersome to derive and list all such restrictions for a given application, these additional testable restrictions are included in our over-identification test as shown in Proposition 6, and therefore the analyst only needs to test the simple condition in Equation (7).