Non-Parametric Identification and Testing of Quantal Response Equilibrium

Online Appendix: Proofs

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Omitted Proofs

Proof of Proposition 1: Since $p^1,\ p^2 \in \mathcal{P}_i(\mathbf{m}_i^1)$, there must exist two values of $\mathbf{m}_{-i}$—denoted as $\mathbf{m}_{-i}^1$ and $\mathbf{m}_{-i}^2$—such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p^1$ and $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^2) = p^2$. Evaluating Equation (5) at these two values leads to the following equations:

$$\begin{aligned} F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p^1] &= \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1), \\ F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^2) = p^2] &= \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^2). \end{aligned} \tag{21}$$

Given that $F_i^{-1}(p^1)$ and $F_i^{-1}(p^2)$ are known by the analyst, the above system is a linear system with two equations and two unknowns (i.e., $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$ and $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)$). Moreover, the fact that $p^1 \neq p^2$ implies that $F_i^{-1}(p^1) \neq F_i^{-1}(p^2)$ and therefore $p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) \neq p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^2)$. Consequently, the rank condition of the system by Equation (21) is satisfied and both $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$ and $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)$ are point identified.

Fix $\mathbf{m}_i$ at $\mathbf{m}_i^1$ and only consider the variations of $\mathbf{m}_{-i}$. Equation (5) then becomes:

$$F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i})] = \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}). \tag{22}$$

Since $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$ and $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)$ have been identified and $p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})$ is known by the analyst, Equation (22) directly identifies $F_i^{-1}(p)$ $\forall p \in \mathcal{P}_i(\mathbf{m}_i^1)$ with the variations provided by $\mathbf{m}_{-i}$. This completes the proof. $\square$

Proof of Proposition 3: Similar as the argument in the proof of Proposition 1, there exists one value $\mathbf{m}_{-i} = \mathbf{m}_{-i}^1$ such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p^1$ given that $p^1 \in \mathcal{P}_i(\mathbf{m}_i^1)$. Evaluating Equation (5) at this realization $(\mathbf{m}_i^1, \mathbf{m}_{-i}^1)$ would imply the following relationship:

$$F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p^1] = \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1). \tag{23}$$

Since $F_i^{-1}(p^1)$ and $p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1)$ are known to the analyst and $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)$ is normalized to one, Equation (23) contains only one unknown $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$. Consequently, this utility difference is identified. Given the identification of the utility differences, Equation (22) then identifies $F_i^{-1}(p)$ $\forall p \in \mathcal{P}_i(\mathbf{m}_i^1)$ due to the exogenous variation of $\mathbf{m}_{-i}$. This completes the proof. $\square$

Proof of Proposition 4: To prove this proposition, it is suffice to prove that $F_i^{-1}(p^1)$ is identified at only one value p^1. The identification of $F_i^{-1}(p)$ $\forall p \neq p^1$ simply follows Proposition 3.

First consider Assumption 6(a) so that $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) = -\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)$. Plugging this relationship into Equation (23), one could obtain the following equation:

$$\begin{aligned} F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = p^1] &= [1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1)] \cdot \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) \\ \Rightarrow F_i^{-1}(p^1) &= 1 - 2p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1). \end{aligned} \tag{24}$$

The second line identifies the value of $F_i^{-1}(p^1)$ and is the result of the normalization by Assumption 5 such that $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) = 1$.

Next, suppose instead that Assumption 6(b) holds. We prove the case that $m_i^1(a_i, a_{-i}) = m_i^2(1 - a_i, a'_{-i})$ $\forall a_i$ and for $a_{-i} = a'_{-i} = 1$. Therefore, we have $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) = -\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} =$

1). The proofs for the other two cases (i.e., $a_{-i} \neq a'_{-i}$ and $a_{-i} = a'_{-i} = 0$) follow a similar argument and are suppressed.

Let us consider $p^1 \in \mathcal{P}_i(\mathbf{m}_i^1) \cap \mathcal{P}_i(\mathbf{m}_i^2)$. As described above, there must exist two games—denoted as $(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})$ and $(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})$—such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) = p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}) = p^1$. When we evaluate these two games, Equation (5) then turns to:

$$
\begin{aligned}
F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) = p^1] &= \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)}) \\
F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}) = p^1] &= \tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}) \\
&= -\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 0) + \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)}).
\end{aligned}
\tag{25}
$$

The last line of Equation (25) follows from the result that $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) = -\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 1)$. Solving Equation (25), one could identify $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) = \frac{F_i^{-1}(p^1) - 1}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})} + 1$ and $\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 0) = \frac{F_i^{-1}(p^1) + 1}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})} - 1$. Next, consider another two games—denoted as $(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})$ and $(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})$—such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)}) = p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)}) = p^2 \in \mathcal{P}_i(\mathbf{m}_i^1) \cap \mathcal{P}_i(\mathbf{m}_i^2)$. Since $\mathcal{P}_i(\mathbf{m}_i^1) \cap \mathcal{P}_i(\mathbf{m}_i^2)$ includes an interval, we could always find such $p^2 \neq p^1$. Evaluating Equation (5) at the above two realizations implies the following equation:

$$
\begin{aligned}
F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)}) = p^2] &= \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)}) \\
F_i^{-1}[p_i(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)}) = p^2] &= -\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) + [\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 0) + \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1)] \cdot p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)}).
\end{aligned}
\tag{26}
$$

Since the terms on the left-hand side of the above two equations are equal, we could equate them and plug in the identified values of $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$ and $\tilde{\pi}_i(\mathbf{m}_i^2, a_{-i} = 0)$. This transformation then identifies the value of $F_i^{-1}(p^1)$ as the following:

$$
F_i^{-1}(p^1) = \frac{2 - \left[\frac{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})} + \frac{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})}\right]}{\frac{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(2)})}{p_{-i}(\mathbf{m}_i^2, \mathbf{m}_{-i}^{2(1)})} - \frac{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(2)})}{p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^{1(1)})}}. \tag{27}
$$

It can be shown that the denominator of Equation (27) equals $\frac{F_i^{-1}(p^2)+1}{F_i^{-1}(p^1)+1} - \frac{F_i^{-1}(p^2)-1}{F_i^{-1}(p^1)-1}$. Therefore, this denominator is different than zero provided that $F_i^{-1}(p^1) \neq F_i^{-1}(p^2)$. Equation (27) then identifies $F_i^{-1}(p^1)$ and completes the proof. $\square$