Non-Parametric Identification and Testing of Quantal Response Equilibrium

Online Appendix: Proofs

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Omitted Proofs

Proof of Equation (5): Recall that $\tilde{\pi}_i(\mathbf{m}_i, a_{-i}) = \pi_i(\mathbf{m}_i, a_i = 1, a_{-i}) - \pi_i(\mathbf{m}_i, a_i = 0, a_{-i})$ denotes the difference of player i's utilities. Plugging the expression of expected payoff $E\pi_i(\cdot)$ by Equation (2) into Equation (4) would imply the following:

$$F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}_{-i})] = \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0) + [\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)]p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i}). \tag{17}$$

Consider two realizations of $\mathbf{m}_{-i}$, say $\mathbf{m}^1_{-i}$ and $\mathbf{m}^2_{-i}$. Plugging them separately into Equation (17) and subtracting them would yield the following equation:

$$\begin{aligned} &F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^2_{-i})] - F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^1_{-i})] \\ =&[\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)] \cdot [p_{-i}(\mathbf{m}_i, \mathbf{m}^2_{-i}) - p_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i})]. \end{aligned} \tag{18}$$

By a similar argument, for realizations $\mathbf{m}^1_{-i}$ and $\mathbf{m}^3_{-i}$, we can derive the following:

$$\begin{aligned} &F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^3_{-i})] - F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^1_{-i})] \\ =&[\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)] \cdot [p_{-i}(\mathbf{m}_i, \mathbf{m}^3_{-i}) - p_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i})]. \end{aligned} \tag{19}$$

Dividing Equation (19) by Equation (18) would yield Equation (5). This completes the proof. □

Proof of Proposition 2: Consider the realization $\mathbf{m}_i = \mathbf{m}_i^1$, Equation (4) turns to the following:

$$F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i})] = \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)]p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}). \tag{20}$$

Note that Equation (20) only considers the variations of $\mathbf{m}_{-i}$. Such variations identify the sign of $[\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)]$. Specifically, the sign is positive (negative) if $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i})$ is increasing (decreasing) in $p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i})$. In addition, the condition that $1/2 \in int[\mathcal{P}_i(\mathbf{m}_i^1)]$ implies the following: There must exist at least one realization $\mathbf{m}_{-i}^1$ such that $p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = 1/2$. Evaluating Equation (20) at this realization implies the following:

$$\begin{aligned}
&\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0) + [\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)]p_{-i}(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) \\
=&F_i^{-1}[p_i(\mathbf{m}_i^1, \mathbf{m}_{-i}^1) = 1/2] \\
=&0.
\end{aligned} \tag{21}$$

The last equality follows Assumption 3(b) such that $F_i(0) = 1/2$. Since $p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i})$ is positive, Equation (21) directly identifies the sign of $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$. Specifically, it equals the negative of the sign of $[\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)]$, which has been identified. Moreover, Assumption 3(a) normalizes $\left|\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)\right|$ to be 1. Together with the identified sign, it identifies the value of $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$.

Since $\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)$ and $p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i})$ are either identified or known, Equation (21) further implies that $[\tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i^1, a_{-i} = 0)]$ is also identified. Consequently, every term on the right hand side of Equation (20) has been either identified or observed. Therefore, Equation (20) directly identifies $F_i^{-1}(p)\ \forall p \in \mathcal{P}_i(\mathbf{m}_i^1)$ with the variations provided by $\mathbf{m}_{-i}$. This completes the proof. $\square$

Proof of Proposition 3: Consider realizations of $\mathbf{m}_{-i} = \mathbf{m}^1_{-i}$, $\mathbf{m}^2_{-i}$. Evaluating Equation (4) under these two realizations implies the following:

$$F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^1_{-i})] = \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0) + [\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)]p_{-i}(\mathbf{m}_i, \mathbf{m}^1_{-i})$$
$$F_i^{-1}[p_i(\mathbf{m}_i, \mathbf{m}^2_{-i})] = \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0) + [\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1) - \tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)]p_{-i}(\mathbf{m}_i, \mathbf{m}^2_{-i}). \tag{22}$$

Since $F_i^{-1}(\cdot)$ has been identified by Proposition 2, Equation (22) is then a linear system with two equations and two unknowns (i.e., $\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 0)$ and $\tilde{\pi}_i(\mathbf{m}_i, a_{-i} = 1)$. The rank condition is satisfied as $p_{-i}(\mathbf{m}_i, \mathbf{m}_{-i})$ varies with $\mathbf{m}_{-i}$. Consequently, the utility difference $\tilde{\pi}_i(\mathbf{m}_i, a_{-i})$ is identified $\forall \mathbf{m}_i,\ a_{-i}$. It completes the proof. $\square$