Non-Parametric Identification and Testing of Quantal Response Equilibrium

Online Appendix: Comparative Statics Analysis

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We consider a transformed random error $\hat{\tilde{\varepsilon}}_i = \frac{1}{\lambda}\tilde{\varepsilon}_i$ that scales up the standard deviation of $\tilde{\varepsilon}_i$ by $\frac{1}{\lambda}$, where $0 < \lambda < \infty$. Under this scaled error, player i's choice probability is characterized as:

$$p_i(\mathbf{m}) = F_i\big\{\lambda\left[EU_i(\mathbf{m}_i, a_i = 0, b_i(\mathbf{m})) - EU_i(\mathbf{m}_i, a_i = 1, b_i(\mathbf{m}))\right]\big\}, \tag{42}$$

where $b_i(\mathbf{m})$ represents player i's belief about the probability that player $-i$ will choose $a_{-i} = 0$. QRE places the restriction that $b_i(\mathbf{m}) = p_{-i}(\mathbf{m})$ so that Equation (42) turns to the quantal response function by Equation (3) when $\lambda = 1$. Moreover, Equation (42) also includes Level-k behaviors when $b_i(\mathbf{m})$ is the belief of the level-k player.

Equation (42) indicates that as λ increases, or equivalently as $Var(\tilde{\varepsilon}_i)$ decreases, player i will choose $a_i = 0$ more (less) frequently if such an action has a higher (lower) expected utility than $a_i = 1$. Furthermore, when $\lambda \to \infty$, player i will unambiguously choose the action that maximizes the expected utility, provided that $F_i(-\infty) = 0$ and $F_i(\infty) = 1$. Conversely, as $\lambda \to 0$, player i will choose $a_i = 0$ with probability $F_i(0)$. If the analyst imposes the restriction that $Median(\tilde{\varepsilon}_i) = 0$ so that $F_i(0) = 1/2$, then player i simply randomizes each action with equal probability.

Next, consider the matching pennies game in Table 1 and suppose that the analyst imposes the QRE restrictions. Given the normalization that the utility of the lowest payoff (i.e., $m = 8$) is zero, each player's $p_i(\mathbf{m})$ is determined by the following equation system:

$$
\begin{aligned}
p_1(m_1,m_2) &= F_1\big\{\lambda\big[(u_1(m_1)+u_1(16))\cdot p_2(m_1,m_2)-u_1(16)\big]\big\},\\
p_2(m_1,m_2) &= F_2\big\{\lambda\big[u_2(m_2)-(u_2(m_2)+u_2(16))\cdot p_1(m_1,m_2)\big]\big\}.
\end{aligned}
\tag{43}
$$

Suppose that both $u_i(m)$ and $F_i(\tilde{\varepsilon}_i)$ are continuously differentiable, then taking derivative with respect to (m_1,m_2) on both sides of Equation (43) would imply the following comparative statics under the QRE framework:

$$
\begin{aligned}
\frac{\partial p_1(\mathbf{m})}{\partial m_1} &= \frac{\lambda\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot u_1'(m_1)\cdot p_2(\mathbf{m})}{1+\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot[u_1(m_1)+u_1(16)]\cdot[u_2(m_2)+u_2(16)]} > 0,\\
\frac{\partial p_1(\mathbf{m})}{\partial m_2} &= \frac{\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot[u_1(m_1)+u_1(16)]\cdot u_2'(m_2)\cdot[1-p_1(\mathbf{m})]}{1+\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot[u_1(m_1)+u_1(16)]\cdot[u_2(m_2)+u_2(16)]} > 0,\\
\frac{\partial p_2(\mathbf{m})}{\partial m_1} &= \frac{-\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot u_1'(m_1)\cdot[u_2(m_2)+u_2(16)]\cdot p_2(\mathbf{m})}{1+\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot[u_1(m_1)+u_1(16)]\cdot[u_2(m_2)+u_2(16)]} < 0,\\
\frac{\partial p_2(\mathbf{m})}{\partial m_2} &= \frac{\lambda\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot u_2'(m_2)\cdot[1-p_1(\mathbf{m})]}{1+\lambda^2\cdot f_1(\lambda\cdot\widetilde{EU}_1)\cdot f_2(\lambda\cdot\widetilde{EU}_2)\cdot[u_1(m_1)+u_1(16)]\cdot[u_2(m_2)+u_2(16)]} > 0,
\end{aligned}
\tag{44}
$$

where $f_i(\cdot)$ is the P.D.F. of $\tilde{\varepsilon}_i$ and $\widetilde{EU}_i$ is the difference between the expected utilities of actions 0 and 1. Moreover, both $p_i(\mathbf{m})$ and $\widetilde{EU}_i$ are evaluated at the QRE conditions. The directions of the own-payoff effect and the other-payoff effect, as shown in Equation (44), are intuitive and are consistent with the reduced form results in Table 6. Moreover, Equation (44) also provides insights into the comparative statics of these effects with respect to λ. When $\lambda \to 0$, both $\frac{\partial p_i(\mathbf{m})}{\partial m_i}$ and $\frac{\partial p_i(\mathbf{m})}{\partial m_{-i}}$ converge to zero. These diminishing own-payoff and other-payoff effects are consistent with the property that each player randomizes each action with equal probability when $\lambda \to 0$. Conversely, consider the other extreme that $\lambda \to \infty$. Since the expression of $\frac{\partial p_i(\mathbf{m})}{\partial m_i}$ has the term λ on its nominator and the term λ^2 in the denominator, the effect of own payoff m_i on $p_i(\mathbf{m})$ decreases in the order of λ. Conversely, the expression of $\frac{\partial p_i(\mathbf{m})}{\partial m_{-i}}$ has the term λ^2 in its both nominator and denominator. Therefore, the effect of the other player's payoff m_{-i} on $p_i(\mathbf{m})$ is order-invariant with respect to λ. As $\lambda \to \infty$, the own-payoff effect disappears while the other-payoff effect remains, as predicted in Nash Equilibrium.

Equation (45) offers another perspective for interpreting the comparative statics of the other-payoff effect. In QRE, player i anticipates that player $-i$'s payoff m_{-i} has a diminishing (in order of λ) on player $-i$'s choice probability $p_{-i}(\mathbf{m})$. This diminishing impact is entirely offset by the effect of $p_{-i}(\mathbf{m})$ on $p_i(\mathbf{m})$, which grows in the order of λ as shown in Equation (45). Consequently, the other-payoff effect, quantified by $\frac{\partial p_i(\mathbf{m})}{m_{-i}}$, is order-invariant with respect to λ.

$$\begin{aligned}\frac{\partial p_1(\mathbf{m})}{\partial p_2(\mathbf{m})} &= \lambda f_1(\lambda \widetilde{EU}_1)[u_1(m_1) + u_1(16)],\\ \frac{\partial p_2(\mathbf{m})}{\partial p_1(\mathbf{m})} &= -\lambda f_2(\lambda \widetilde{EU}_1)[u_2(m_2) + u_2(16)].\end{aligned} \tag{45}$$

The structure of matching pennies game in Table 1 also implies an interesting feature under Level-k behaviors. Specifically, when $m_i < 16$ ($m_i > 16$), the level-1 player would obtain a strictly lower (higher) expected utility of action 0 than action 1. Therefore, as $\lambda \to \infty$, the level-1 player will choose $a_i = 0$ with probability 0 (1). Due to the hierarchy of beliefs, players with higher types would also choose one of the actions with certainty, and such a choice is independent of players' risk preference. In summary, under level-k models, the effect of players' risk preference parameter ν on their behaviors vanishes in the limiting case as $\lambda \to \infty$ or $\lambda \to 0$.[25]

Figures 11 to 14 plot $p_i(m_i, m_{-i})$ for both players in our Monte Carlo exercise. These figures aim to illustrate how the value of $Var(\tilde{\varepsilon}_i)$ will affect each player's behavior under various models, including QRE and Level-k with $k \in \{1,2,3\}$. We consider three scenarios: (1) original value of $Var(\tilde{\varepsilon}_i)$ in our Monte Carlo exercise, (2) doubling the value of

[25] Note that when $\lambda \to 0$, each player randomizes their actions with equal probability, regardless of their expected utilities.

$Var(\tilde{\varepsilon}_i)$, and (3) the limiting case where $Var(\tilde{\varepsilon}_i) \to 0$. Clearly, these figures demonstrate the substantial impact of $Var(\tilde{\varepsilon}_i)$ on each player's behavior.

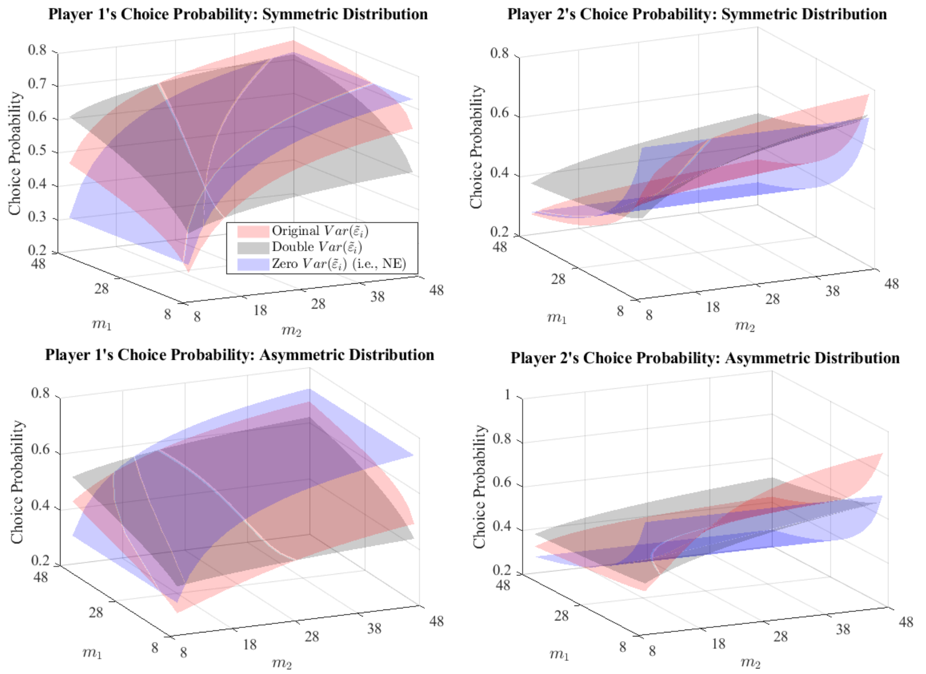

Figure 11: Players' Choice Probabilities: QRE Behavior

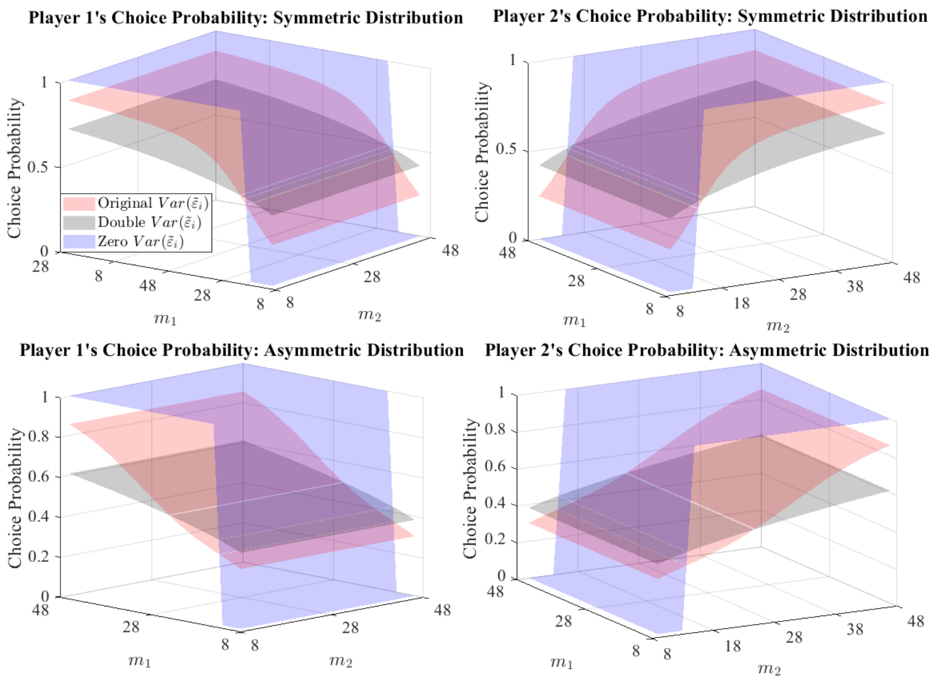

Figure 12: Players' Choice Probabilities: Level-1 Reasoning Behavior

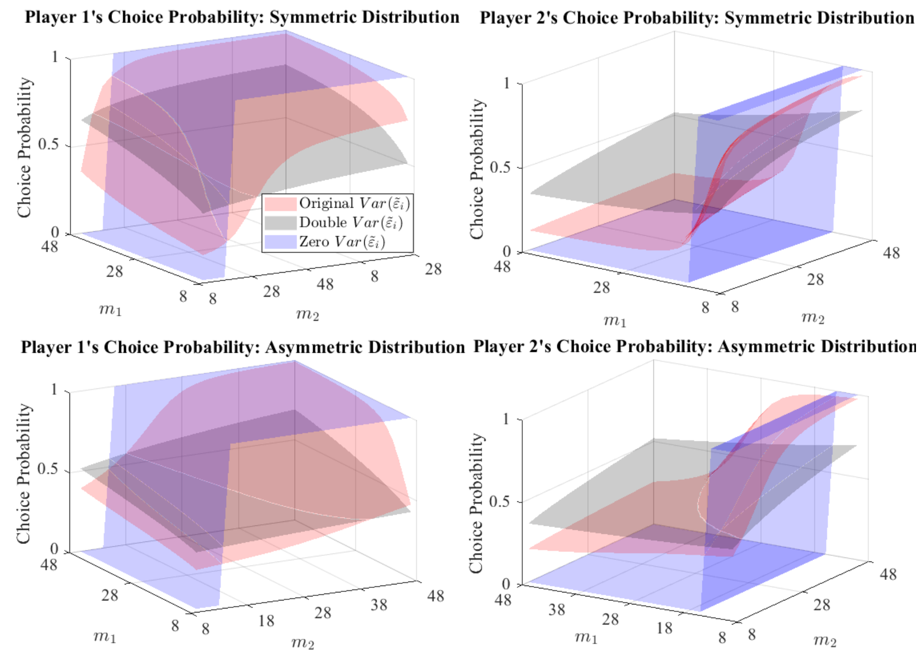

Figure 13: Players' Choice Probabilities: Level-2 Reasoning Behavior

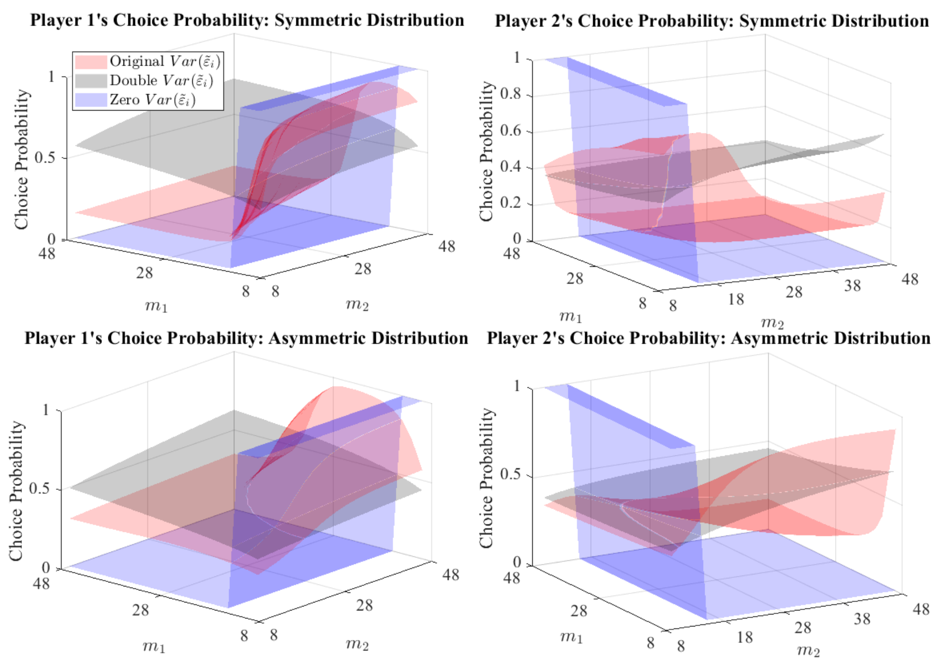

Figure 14: Players' Choice Probabilities: Level-3 Reasoning Behavior